A Coarea Formula for Smooth Contact Mappings of Carnot–Carathéodory Spaces

M. Karmanova, S. Vodopyanov[*]

October 26, 2011

Abstract

We prove the coarea formula for sufficiently smooth contact mappings of Carnot manifolds. In particular, we investigate level surfaces of these mappings, and compare Riemannian and sub-Riemannian measures on them. Our main tool is the sharp asymptotic behavior of the Riemannian measure of the intersection of a tangent plane to a level surface and a sub-Riemannian ball. This calculation in particular implies that the sub-Riemannian measure of the set of characteristic points (i. e., the points at which the sub-Riemannian differential is degenerate) equals zero on almost every level set.

Contents

1 Introduction

This article is devoted to a sub-Riemannian analog

$$\int_{\mathbb{M}} \mathcal{J}_{\tilde{N}}^{SR}(\varphi, x) \, d\mathcal{H}^{\nu}(x) = \int_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int_{\varphi^{-1}(t)} d\mathcal{H}^{\nu - \tilde{\nu}}(u) \qquad (1.1)$$

of the well-known coarea formula (1.2). Here $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ is a smooth contact mapping of Carnot–Carathéodory spaces, and $\mathcal{J}_{\tilde{N}}^{SR}(\varphi, x)$ is a sub-Riemannian coarea factor determined by the values of a sub-Riemannian differential, or *hc-differential*. In some sense, we may regard (1.1) as a generalization of the results of [Pa, He, Mg1, Mg2] and some other articles.

It is well known that the coarea formula

$$\int_{U} \mathcal{J}_{k}(\varphi, x) \, dx = \int_{\mathbb{R}^{k}} dz \int_{\varphi^{-1}(z)} d\mathcal{H}^{n-k}(u), \qquad (1.2)$$

where $\mathcal{J}_{k}(\varphi, x) = \sqrt{\det(D\varphi(x) D\varphi^{*}(x))}$, has many applications in analysis on Euclidean spaces [Kr, Fe1, Fe2, EG, GMS, LiYa, Ot]. Here we assume that $\varphi \in C^{1}(U, \mathbb{R}^{k})$ with $U \subset \mathbb{R}^{n}$ for $n \geq k$. In particular, (1.2) applies in the theory of exterior forms and currents, and in problems about minimal surfaces (see, for example, [FeFl]). Also, Stokes' formula easily follows from the coarea formula (see, for instance, [Vod1]). The development of analysis on more general structures raises a natural question of extending the coarea formula to objects with a more general geometry than in Euclidean spaces, especially to metric spaces and sub-Riemannian manifolds. In 1999, L. Ambrosio and B. Kirchheim [AmKi] proved an analog of the coarea formula for Lipschitz mappings defined on an $\mathcal{H}^{n}$-rectifiable metric space with values in $\mathbb{R}^{k}$ for $n \geq k$. In 2004, this formula was established for Lipschitz mappings defined on an $\mathcal{H}^{n}$-rectifiable metric space with values in an $\mathcal{H}^{k}$-rectifiable metric space for $n \geq k$ [Km1, Km4]. Moreover, conditions were found on the image and preimage of a Lipschitz mapping defined on an $\mathcal{H}^{n}$-rectifiable metric space with values in an *arbitrary* metric space which are necessary and sufficient for the validity of the coarea formula. Independently of that result, the level sets of those mappings were investigated, and the metric analog of implicit function theorem was proved [Km2, Km4, Km6].

All results mentioned above were obtained on rectifiable metric spaces. Note that their metric structure is similar to that of Riemannian manifolds. However, there exist *non-rectifiable* metric spaces whose geometry is not comparable to the Riemannian one. *Carnot manifolds* are of special interest (for results on their non-rectifiability, see, for instance, [AmKi]). Sub-Riemannian

geometry naturally arises in the theory of subelliptic equations, contact geometry, optimal control theory, non-holonomic mechanics, neurobiology, and other areas (see, e. g., [NSW, Gr, Be, Mg1, MrMo1, MrMo2, AgMa, Je, VodG, Vod2, Vod3, CS, HlPa]). This theory has many applications. In addition, it has many well-known unsolved problems.

One of them is the problem of the sub-Riemannian coarea formula, which is useful for developing a non-holonomic theory of currents, exterior forms, extremal surfaces (in sub-Riemannian and sub-Lorentzian geometries), and so forth.

Heisenberg groups and Carnot groups are well-known particular cases of Carnot manifolds. In 1982, P. Pansu proved the coarea formula for real-valued functions defined on a Heisenberg group [Pa]. Next, J. Heinonen [He] extended this formula to smooth functions defined on a Carnot group. Another result concerning the analog of (1.2) belongs to V. Magnani. In 2000, he proved a *coarea inequality* for mappings of Carnot groups [Mg1]. The equality was established for a mapping from a Heisenberg group to the Euclidean space $\mathbb{R}^k$ [Mg2]. The validity of coarea formula even for a model case of a mapping from a Carnot group to other Carnot group has remained an open question.

The purpose of this paper is to prove the coarea formula for sufficiently smooth contact mappings $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ of Carnot manifolds. We emphasize that all results are new even in the particular case of a mapping of Carnot groups.

As we mentioned above, for the first time a non-holonomic analogue of the coarea formula was discovered by P. Pansu [Pa]. One of the basic ideas was to prove the sub-Riemannian coarea formula via the Riemannian one:

$$(1.2) \Rightarrow \int_U \mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x) \, d\mathcal{H}^{\nu}(x)$$

$$= \int_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(z) \int_{\varphi^{-1}(z)} \frac{\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, u)}{\mathcal{J}_{\widetilde{N}}(\varphi, x)} \, d\mathcal{H}^{N-\widetilde{N}}(u) \overset{?}{=} \int_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(z) \int_{\varphi^{-1}(z)} d\mathcal{H}^{\nu - \tilde{\nu}}(u)$$

$$(1.3)$$

Here N and $\widetilde{N}$ are the topological dimensions while ν and $\tilde{\nu}$ are the Hausdorff dimensions of the preimage and image respectively. It is clear that in sub-Riemannian spaces these dimensions differ. Other authors subsequently used the same idea.

It follows easily from (1.3) that the crucial point in this method is to understand the relation between the Riemannian and sub-Riemannian measures on the Carnot manifolds themselves and on the level sets. Moreover,

an appropriate definition of the sub-Riemannian coarea factor is required. It is well known that the question of measure relation on the given spaces is trivial, whereas those of the geometry of level sets and the sub-Riemannian coarea factor are quite complicated. The main difficulties are related to the peculiarities of sub-Riemannian metrics. The inequivalence of Riemannian and sub-Riemannian metrics is illustrated for example by the fact that the Riemannian distance from the center of a radius r sub-Riemannian ball to its boundary varies from r to r^M, $M > 1$, where the constant M depends on the structure of the space. Thus, an immediate question arises: how "sharply" does the tangent plane approximate a level set? (This question is important since the Riemannian tangency order $o(r)$ is insufficient here: a level surface can "jump" out of a sub-Riemannian ball much earlier than necessary if this order of tangency is higher than $o(r)$.) Also, it is a problem whether or not there exists a metric allowing us to describe the shape and geometry of the intersection of a sub-Riemannian ball and a level surface. Even if we answer these two questions, the most difficult question concerns the relation between the Hausdorff dimensions of the image and the intersection of a ball and a level set.

In this article we solve problems stated above. Firstly, we divide the set of all points at which the classical differential is non-degenerate into a regular set and a characteristic set. Next, we define a sub-Riemannian quasimetric d_2 that enables us to calculate the measure of the intersection of a sub-Riemannian ball and a tangent plane (see below Remark 3.8 for details). A crucial idea in the construction of d_2 is based on the fact that a ball in this quasimetric is asymptotically equal to direct product of Euclidean balls:

$$B_{d_2}^N(x, r) \approx B^{n_1}(x, r) \times B^{n_2}(x, r^2) \times \ldots \times B^{n_M}(x, r^M), \ M > 1,$$

where N is the topological dimension of a Carnot manifold, and n_i are the (topological) dimensions of the Euclidean balls B^{n_i}, $i = 1, \ldots, M$. Thus, if a plane intersects a ball of this type then we can easily determine the shape of the intersection (contrary to the case of a "box" quasimetric with

$$\mathrm{Box}(x, r) \approx Q^{n_1}(x, r) \times Q^{n_2}(x, r^2) \times \ldots \times Q^{n_M}(x, r^M),$$

where we have Euclidean cubes Q^{n_i}, $i = 1, \ldots, M$, instead of balls, since the cubes have sections of different shapes). Studying the sharpness of the approximation to a level surface by its tangent plane, we introduce a "mixed" quasimetric possessing both Riemannian and sub-Riemannian properties. We prove that at regular points the tangent plane approximates the level surface sufficiently well, and this fact enables us to calculate the (Riemannian)

measure of this intersection, which depends on the Hausdorff dimensions of both the image and preimage. In other words, it is equivalent to $r^{\nu - \tilde{\nu}}$ (see Theorems 3.7 and 3.11). These results yield analytic expression of the relation between the Riemannian and sub-Riemannian measures at the regular points on level sets (see Theorem 3.17).

The characteristic set case is a little more complicated since precisely near a characteristic point a surface can "jump" out of a sub-Riemannian ball. For this reason we cannot estimate the measure of the intersection of a ball and a surface via the measure of the intersection of a ball and the tangent plane at a characteristic point. Note also that in all papers mentioned above the preimage has a group structure, whose properties are essential for proving that the measure of the set of all characteristic points on each level surface equals zero. In the case of a mapping of two Carnot manifolds, both the image and preimage lack a group structure, and the approximation of a manifold by its local Carnot group is insufficient for the extension of the methods developed to our case. That is why we create an "intrinsic" method for studying the properties of the characteristic set.

The result on the characteristic set is stated in Theorem 4.1.

In Section 4 we prove that the degenerate set of the classical differential does not affect either part of the coarea formula.

Finally, in Section 5, we define the sub-Riemannian coarea factor (via the values of the hc-differential) and derive the sub-Riemannian coarea formula (1.1).

For mappings $\varphi : \mathbb{H}^1 \to \mathbb{R}$, formula (1.1) becomes

$$\int\limits_{\mathbb{M}} |\nabla_H \varphi(x)| \cdot \frac{\omega_3^2}{\omega_4 \cdot 4} \, d\mathcal{H}^\nu(x) = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t)} d\mathcal{H}^{\nu - \tilde{\nu}}(u)$$

since in this case we have

$$\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x) = |\nabla_H \varphi(x)| \cdot \frac{\omega_3^2}{\omega_4 \cdot 4}.$$

Here $\nabla_H \varphi(x)$ is the "horizontal part" of the gradient of φ at x, and the coefficient $\frac{\omega_3^2}{\omega_4 \cdot 4}$ appears due to the choice of a constant factor in the definition of the Hausdorff measure (to define $\mathcal{H}^\mu$ we set this factor to ω_μ, and if we consider 2^μ instead of ω_μ then this factor will be equal to 1); cf. [Pa]. For

$\varphi : \mathbb{H}^n \to \mathbb{R}^k$, $k \leq 2n$, we deduce that

$$\int\limits_{\mathbb{M}} \sqrt{\det(D_H\varphi(x)D_H\varphi(x)^*)} \cdot \frac{\omega_{2n+1} \cdot \omega_{2n+2-k}}{\omega_{2n+2} \cdot \omega_{2n-k} \cdot 2} \, d\mathcal{H}^\nu(x)$$

$$= \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t)} d\mathcal{H}^{\nu-\tilde{\nu}}(u)$$

and

$$\mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) = \sqrt{\det(D_H\varphi(x)D_H\varphi(x)^*)} \cdot \frac{\omega_{2n+1} \cdot \omega_{2n+2-k}}{\omega_{2n+2} \cdot \omega_{2n-k} \cdot 2}.$$

Here $D_H\varphi(x)$ is the "horizontal part" of the differential $D\varphi(x)$, and if we replace all occurrences of ω_μ by 2^μ then the coefficient following the Gram determinant of $D_H\varphi(x)$ will be equal to 1; see also [Mg2]. Similarly, for $\mathbb{M} = \mathbb{G}$ and $\widetilde{\mathbb{M}} = \mathbb{R}$ we obtain

$$\int\limits_{\mathbb{M}} |\nabla_H\varphi(x)| \cdot \frac{\omega_N \cdot \omega_{\nu-1}}{\omega_\nu \cdot \prod\limits_{k=2}^{M} \omega_{n_k} \cdot \omega_{n_1-1}} \, d\mathcal{H}^\nu(x) = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t)} d\mathcal{H}^{\nu-\tilde{\nu}}(u)$$

and

$$\mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) = |\nabla_H\varphi(x)| \cdot \frac{\omega_N \cdot \omega_{\nu-1}}{\omega_\nu \cdot \prod\limits_{k=2}^{M} \omega_{n_k} \cdot \omega_{n_1-1}}.$$

Once again, it is easy to see that if we take 2^μ instead of ω_μ then the coefficient following the length of the horizontal gradient of φ at x will be equal to 1; compare with [He].

Thus, for sufficiently smooth mappings we obtain all previous results as particular cases.

2   Preliminaries

In this section we introduce some necessary definitions and mention important facts used to prove the main result.

Definition 2.1 (cf. [Gr, KmVod, NSW])**.** Fix a connected Riemannian C^∞-manifold $\mathbb{M}$ of topological dimension N. The manifold $\mathbb{M}$ is called a *Carnot–Carathéodory space* if the tangent bundle $T\mathbb{M}$ has a filtration

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \ldots \subsetneq H_i\mathbb{M} \subsetneq \ldots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles such that each point $p \in \mathbb{M}$ has a neighborhood $U \subset \mathbb{M}$ equipped with a collection of $C^{1,\alpha}$-smooth vector fields $X_1, \ldots, X_N$, $\alpha \in (0,1]$, enjoying the following two properties. For each $v \in U$,

(1) $H_i \mathbb{M}(v) = H_i(v) = \mathrm{span}\{X_1(v), \ldots, X_{\dim H_i}(v)\}$ is a subspace of $T_v \mathbb{M}$ of a constant dimension $\dim H_i$, $i = 1, \ldots, M$;

(2) we have

$$[X_i, X_j](v) = \sum_{k:\, \deg X_k \leq \deg X_i + \deg X_j} c_{ijk}(v) X_k(v) \qquad (2.1)$$

where the *degree* $\deg X_k$ is defined as $\min\{m \mid X_k \in H_m\}$;

Moreover, if the third condition holds then the Carnot–Carathéodory space is called the *Carnot manifold*:

(3) the quotient mapping $[\cdot, \cdot]_0 : H_1 \times H_j / H_{j-1} \mapsto H_{j+1} / H_j$ induced by the Lie bracket is an epimorphism for all $1 \leq j < M$.

The subbundle $H\mathbb{M}$ is called *horizontal*.

The number M is called the *depth* of the manifold $\mathbb{M}$.

Properties of Carnot-Carathéodory spaces and Carnot manifolds under assumptions of regularity mentioned in definition 2.1 can be found in [KmVod, Km8, Km9, VodKm2, Km10]

Example. A Carnot group is an example of a Carnot manifold.

Definition 2.2. Consider the initial value problem

$$\begin{cases} \dot{\gamma}(t) = \sum\limits_{i=1}^{N} y_i X_i(\gamma(t)), \ t \in [0,1], \\ \gamma(0) = x, \end{cases}$$

where the vector fields $X_1, \ldots, X_N$ are C^1-smooth. Then, for the point $y = \gamma(1)$ we write $y = \exp\left(\sum\limits_{i=1}^{N} y_i X_i\right)(x)$.

The mapping $(y_1, \ldots, y_N) \mapsto \exp\left(\sum\limits_{i=1}^{N} y_i X_i\right)(x)$ is called the *exponential*.

Definition 2.3. Consider $u \in \mathbb{M}$ and $(v_1, \ldots, v_N) \in B_E(0, r)$, where $B_E(0, r)$ is a Euclidean ball in $\mathbb{R}^N$. Define a mapping $\theta_u : B_E(0, r) \to \mathbb{M}$ as follows:

$$\theta_u(v_1, \ldots, v_N) = \exp\left(\sum_{i=1}^{N} v_i X_i\right)(u).$$

It is known that θ_u is a C^1-diffeomorphism if $0 < r \leq r_u$ for some $r_u > 0$. The collection $\{v_i\}_{i=1}^{N}$ is called *the normal coordinates* or *the coordinates of the 1st kind* (*with respect to $u \in \mathbb{M}$*) of the point $v = \theta_u(v_1, \ldots, v_N)$.

Theorem 2.4 ([KmVod]). *Fix $u \in \mathbb{M}$. The coefficients*

$$\bar{c}_{ijk} = \begin{cases} c_{ijk}(u) \ of \ (2.1) & if \ \deg X_i + \deg X_j = \deg X_k, \\ 0 & otherwise \end{cases}$$

define a graded nilpotent Lie algebra.

We construct the Lie algebra $\mathfrak{g}^u$ of Theorem 2.4 as a graded nilpotent Lie algebra of vector fields $\{(\widehat{X}_i^u)'\}_{i=1}^N$ on $\mathbb{R}^N$ such that the exponential mapping $(x_1, \ldots, x_N) \mapsto \exp\left(\sum_{i=1}^N x_i(\widehat{X}_i^u)'\right)(0)$ is the identity [Po, BLU]. In view of the results of [FoSt], the value of $(\widehat{X}_j^u)'(0)$ is equal to a standard vector $e_{i_j} \in \mathbb{R}^N$, where $i_j \neq i_k$ if $j \neq k$, $j = 1, \ldots, N$. We associate to each vector field of the resulting collection an index i so that $(\theta_u)_* \langle (\widehat{X}_i^u)' \rangle (u) = X_i(u)$. By the construction, the vector fields $\{(\widehat{X}_i^u)'\}_{i=1}^N$ satisfy

$$[(\widehat{X}_i^u)', (\widehat{X}_j^u)'] = \sum_{\deg X_k = \deg X_i + \deg X_j} c_{ijk}(u)(\widehat{X}_k^u)' \tag{2.2}$$

everywhere on $\mathbb{R}^N$.

Notation 2.5. We use the following standard notation: for each N-dimensional multi-index $\mu = (\mu_1, \ldots, \mu_N)$, its *homogeneous norm* equals $|\mu|_h = \sum_{i=1}^N \mu_i \deg X_i$.

Definition 2.6. Refer as the graded nilpotent Carnot group $\mathbb{G}_u \mathbb{M}$ corresponding to the Lie algebra $\mathfrak{g}^u$ to the *nilpotent tangent cone* of $\mathbb{M}$ at $u \in \mathbb{M}$. We construct $\mathbb{G}_u \mathbb{M}$ in $\mathbb{R}^N$ as a groupalgebra [Po], that is, the exponential map is the identity:

$$\exp\left(\sum_{i=1}^N x_i(\widehat{X}_i^u)'\right)(0) = (x_1, \ldots, x_N).$$

By the Baker–Campbell–Hausdorff formula, the group operation is defined so that the basis vector fields $(\widehat{X}_i^u)'$ on $\mathbb{R}^N$, $i = 1, \ldots, N$, are left-invariant [Po]: if

$$x = \exp\left(\sum_{i=1}^N x_i(\widehat{X}_i^u)'\right), \ y = \exp\left(\sum_{i=1}^N y_i(\widehat{X}_i^u)'\right)$$

then

$$x \cdot y = z = \exp\left(\sum_{i=1}^N z_i(\widehat{X}_i^u)'\right),$$

where

$$z_i = x_i + y_i, \quad \deg X_i = 1,$$

$$z_i = x_i + y_i + \sum_{\substack{|e_l+e_j|_h=2, \\ l<j}} F^i_{e_l,e_j}(u)(x_l y_j - y_l x_j), \quad \deg X_i = 2,$$

$$z_i = x_i + y_i + \sum_{\substack{|\mu+\beta|_h=k, \\ \mu>0, \, \beta>0}} F^i_{\mu,\beta}(u) x^\mu \cdot y^\beta \qquad (2.3)$$

$$= x_i + y_i + \sum_{\substack{|\mu+e_l+\beta+e_j|_h=k, \\ l<j}} G^i_{\mu,\beta,l,j}(u) x^\mu y^\beta (x_l y_j - y_l x_j), \quad \deg X_i = k.$$

Using the exponential mapping θ_u, we can push forward the vector fields $(\widehat{X}^u_i)'$ onto $\mathcal{U} \subset \mathbb{M}$ as

$$[(\theta_u)_* \langle (\widehat{X}^u_i)' \rangle](\theta_u(x)) = D\theta_u(x) \langle (\widehat{X}^u_i)'(x) \rangle$$

and obtain the vector fields $\widehat{X}^u_i = (\theta_u)_* (\widehat{X}^u_i)'$. Recall that $\widehat{X}^u_i(u) = X_i(u)$.

Definition 2.7. Associated to the Lie algebra $\{\widehat{X}^u_i\}_{i=1}^N$ at $u \in \mathbb{M}$, is a *local homogeneous group* $\mathcal{G}^u\mathbb{M}$. Define it so that the mapping θ_u is a *local group isomorphism* between some neighborhoods of the identity elements of the groups $\mathbb{G}_u\mathbb{M}$ and $\mathcal{G}^u\mathbb{M}$.

The canonical Riemannian structure on $\mathcal{G}^u\mathbb{M}$ is determined by the inner product at the identity element of $\mathcal{G}^u\mathbb{M}$ coinciding with that on $T_u\mathbb{M}$. The canonical Riemannian structure on the nilpotent tangent cone $\mathbb{G}_u\mathbb{M}$ is defined so that the local group isomorphism θ_u is an isometry.

Assumption 2.8. Henceforth we assume that the neighborhood $\mathcal{U}$ under consideration is such that $\mathcal{U} \subset \mathcal{G}^u\mathbb{M}$ for all $u \in \mathcal{U}$.

Definition 2.9. A curve $\gamma : [0,1] \to \mathbb{M}$ which is absolutely continuous in the Riemannian sense is called *horizontal* if $\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0,1]$ with respect to the Lebesgue measure on $[0,1]$.

A horizontal curve in $\mathcal{G}^u\mathbb{M}$ is defined similarly: here we require that $\dot{\gamma}(t) \in \operatorname{span}\{\widehat{X}^u_1(\gamma(t)), \ldots, \widehat{X}^u_{\dim H_1}(\gamma(t))\} = \widehat{H}^u_{\gamma(t)}\mathbb{M}$ for almost all $t \in [0,1]$ with respect to the Lebesgue measure on $[0,1]$.

Theorem 2.10 (see [2, 1] for smooth case and [KmVod, Km9] for $C^{1,\alpha}$-smooth case). *Every two points of $\mathbb{M}$ can be joined by a horizontal curve.*

Definition 2.11. Given $x, y \in \mathbb{M}$, the Carnot–Carathéodory distance $d_{cc}(x, y)$ is defined as

$$d_{cc}(x, y) = \inf\{\ell(\gamma) : \gamma : [0, 1] \to \mathbb{M}, \ \dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{M}\},$$

where the length ℓ of each (horizontal) curve γ is calculated with respect to the Riemannian tensor on $\mathbb{M}$.

For $x, y \in \mathcal{G}^u\mathbb{M}$ the Carnot–Carathéodory distance $d_{cc}^u(x, y)$ is defined as

$$d_{cc}^u(x, y) = \inf\{\ell^u(\gamma) : \gamma : [0, 1] \to \mathcal{G}^u\mathbb{M}, \ \dot{\gamma}(t) \in \widehat{H}_{\gamma(t)}^u\mathbb{M}\},$$

where the length ℓ^u of each (horizontal) curve γ is calculated with respect to the Riemannian tensor on $\mathcal{G}^u\mathbb{M}$.

Denote the ball of radius r in d_{cc} centered at x by $B_{cc}(x, r)$. Denote the ball of radius r in d_{cc}^u centered at x by $B_{cc}^u(x, r)$.

Assumption 2.12. Henceforth we assume that $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ is a mapping of Carnot manifolds $\mathbb{M}$ and $\widetilde{\mathbb{M}}$. We specify its smoothness below in Assumption 2.26.

Notation 2.13. Hereinafter, we use the following notation. Namely, we denote:
- the topological dimension of $\mathbb{M}$ ($\widetilde{\mathbb{M}}$) by N ($\widetilde{N}$);
- the Hausdorff dimension of $\mathbb{M}$ ($\widetilde{\mathbb{M}}$) by ν ($\tilde{\nu}$);
- the depth of $\mathbb{M}$ ($\widetilde{\mathbb{M}}$) by M ($\widetilde{M}$).

In addition, we consider
- horizontal subbundles $H = H_1 \subset T\mathbb{M}$ and $\widetilde{H} = \widetilde{H}_1 \subset T\widetilde{\mathbb{M}}$ on $\mathbb{M}$ ($\widetilde{\mathbb{M}}$);
- subspaces $H \subset H_2 \subset \ldots \subset H_M = T\mathbb{M}$ ($\widetilde{H} \subset \widetilde{H}_2 \subset \ldots \subset \widetilde{H}_{\widetilde{M}} = T\widetilde{\mathbb{M}}$) of dimensions $n = n_1 = \dim H_1 < \dim H_2 < \ldots < \dim H_M = N$ ($\tilde{n} = \tilde{n}_1 = \dim \widetilde{H}_1 < \dim \widetilde{H}_2 < \ldots < \dim \widetilde{H}_{\widetilde{M}} = \widetilde{N}$) respectively at every point $x \in \mathbb{M}$ ($\tilde{x} \in \widetilde{\mathbb{M}}$) (see Definition 2.1).

We put $n_1 = \dim H_1$, $\tilde{n}_1 = \dim \widetilde{H}_1$, $n_k = \dim H_k - \dim H_{k-1}$ ($\tilde{n}_k = \dim \widetilde{H}_k - \dim \widetilde{H}_{k-1}$), $k = 2, \ldots, M$ ($\widetilde{M}$).

Denote by d_{cc} ($\widetilde{d}_{cc}$) the Carnot–Carathéodory metric in $\mathbb{M}$ ($\widetilde{\mathbb{M}}$), and by d_{cc}^u ($\widetilde{d}_{cc}^w$) the Carnot–Carathéodory metric in $\mathcal{G}^u\mathbb{M}$ ($\mathcal{G}^w\widetilde{\mathbb{M}}$).

Theorem 2.14 ([Vod4])**.** *Suppose that $E \subset \mathbb{M}$ is an open set, and let $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ be a mapping with C^1-smooth horizontal derivatives $X_i\varphi$ such that $X_i\varphi \in \widetilde{H}$, $i = 1, \ldots, n$. Then, it is hc-differentiable in points of E. Namely, for a point $u \in E$, there exists a horizontal homomorphism $L_u : (\mathcal{G}^u\mathbb{M}, d_{cc}^u) \to (\mathcal{G}^{\varphi(u)}\widetilde{\mathbb{M}}, \widetilde{d}_{cc}^{\varphi(u)})$ of local Carnot groups such that it is continuous in u and*

$$\widetilde{d}_{cc}(\varphi(w), L_u[w]) = o(d_{cc}(u, w)) \ as \ E \cap \mathcal{G}^u\mathbb{M} \ni w \to u$$

where $o(\cdot)$ is locally uniform.

Remark 2.15. Using the exponential mapping θ_u, we can consider L_u both as a homomorphism of local Carnot groups and as a homomorphism of Lie algebras of these local Carnot groups.

Notation 2.16. Henceforth we denote the *hc*-differential L_u of φ at u by the symbol $\widehat{D}\varphi(u)$.

Corollary 2.17 ([Vod4])**.** *Let $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ be a contact (i. e., $D\varphi\langle H \rangle \subset \widetilde{H}$) C^1-mapping of Carnot manifolds (in the Riemannian sense). Then, it is continuously hc-differentiable everywhere on $\mathbb{M}$.*

Property 2.18. Suppose that $\varphi \in C^1(\mathbb{M}, \widetilde{\mathbb{M}})$. Then, the matrices of its differential $D\varphi(u)$ (in the bases $\{X_i(u)\}_{i=1}^N$ and $\{\widetilde{X}_i(\varphi(u))\}_{i=1}^{\widetilde{N}}$) and of its *hc*-differential $\widehat{D}\varphi(u)$ (in the bases $\{\widehat{X}_i^u(v)\}_{i=1}^N$ and $\{\widehat{\widetilde{X}}_i(\varphi(v))\}_{i=1}^{\widetilde{N}}$, where v need not be equal to u) have the following structures:

$$
D\varphi(u) = \begin{pmatrix}
\mathcal{V}_1(u) & * & * & * & * & * & * & \ldots & * \\
0 & \mathcal{V}_2(u) & * & * & * & * & * & \ldots & * \\
0 & 0 & \mathcal{V}_3(u) & * & * & * & * & \ldots & * \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & * & \ldots & * \\
0 & \ldots & 0 & \vdots & \mathcal{V}_{\widetilde{M}-1}(u) & * & * & \ldots & * \\
0 & \ldots & 0 & \ldots & 0 & \mathcal{V}_{\widetilde{M}}(u) & * & \ldots & *
\end{pmatrix},
$$
$$\tag{2.4}$$

$$
\widehat{D}\varphi(u) = \begin{pmatrix}
\mathcal{V}_1(u) & 0 & & \cdots & & 0 & 0 & 0 & \ldots & 0 \\
0 & \mathcal{V}_2(u) & 0 & \cdots & & \vdots & 0 & 0 & \ldots & 0 \\
0 & 0 & \mathcal{V}_3(u) & 0 & & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & & \vdots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \vdots & \mathcal{V}_{\widetilde{M}-1}(u) & & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 & \mathcal{V}_{\widetilde{M}}(u) & 0 & \ldots & 0
\end{pmatrix},
$$
$$\tag{2.5}$$

where each block $\mathcal{V}_i$ is of size $(n_i \times \tilde{n}_i)$, $i = 1, \ldots, M$. Note that, the blocks $\mathcal{V}_i$ are the same in the matrices of $D\varphi(u)$ and $\widehat{D}\varphi(u)$.

Assumption 2.19. Throughout the article we assume that:

- $N \geq \widetilde{N}$;

- If the matrix of $\widehat{D}\varphi(x)$ has full rank at at least one point then we suppose that $n \geq \tilde{n}$. Otherwise, we suppose that $n_i \geq \tilde{n}_i$, $i = 1, \ldots, M$ (see Proposition 2.21).

Notation 2.20. Given $\varphi \in C^1(\mathbb{M}, \widetilde{\mathbb{M}})$, denote by Z the set of points $x \in \mathbb{M}$ with $\mathrm{rank}(D\varphi(x)) < \widetilde{N}$.

Proposition 2.21. I. *Given $\varphi \in C^1(\mathbb{M}, \widetilde{\mathbb{M}})$, consider a point $x \in \mathbb{M} \setminus Z$.*
    (a) If $\widehat{D}\varphi(x)(V_1) = \widetilde{V}_1$ then $\widehat{D}\varphi(x)(V_i) = \widetilde{V}_i$ for all $i = 1, \ldots, M$, and $\mathrm{rank}\,\widehat{D}\varphi(x) = \widetilde{N}$.
    (b) If $\widehat{D}\varphi(x)(V_1) \subsetneq \widetilde{V}_1$ then $\mathrm{rank}\,\widehat{D}\varphi(x) < \widetilde{N}$.
    II. *At the points $x \in \mathbb{M} \setminus Z$ with $\mathrm{rank}\,\widehat{D}\varphi(x) = \widetilde{N}$ we have*
    (a) $D\varphi(x)(H_i) = \widetilde{H}_i$, $i = 1, \ldots, \widetilde{M}$;
    (b) $\mathcal{D}\varphi(x)(H_i/H_{i-1}) = \widetilde{H}_i/\widetilde{H}_{i-1}$, $i = 2, \ldots, \widetilde{M}$. Here for every element $\mathcal{Y} \in H_i/H_{i-1}$ *there exists $Y \in H_i$ such that*

$$\mathcal{Y} = \{Y + T : T \in H_{i-1}\} = Y + H_{i-1},$$

and we define for $\mathcal{Y}$ the value $\mathcal{D}\varphi\langle \mathcal{Y} \rangle$ as the class

$$\mathcal{D}\varphi\langle \mathcal{Y} \rangle = \{D\varphi\langle Y \rangle + D\varphi\langle T \rangle : Y \in H_i \text{ is fixed}, T \in H_{i-1}\}$$
$$= D\varphi\langle Y \rangle + D\varphi\langle H_{i-1} \rangle.$$

Proof. **I. (a)** Fix $x \in \mathbb{M} \setminus Z$ such that $\widehat{D}\varphi(x)(V_1) = \widetilde{V}_1$ and denote $\widehat{D}\varphi(x)$ by L. Verify that $LV_2 = \widetilde{V}_2$. By taking into account property (3) of Definition 2.1 and the property $L[X, Y] = [LX, LY]$ of a group homomorphism L, we infer that $L(V_2) \subset \widetilde{V}_2$. Verify that $\widetilde{V}_2 \subset L(V_2)$. By property (4) of Definition 2.1, for each element $\widetilde{Y} \in \widetilde{V}_2$ we have $\widetilde{Y} = [\widetilde{Y}_1, \widetilde{Y}_2]$, where $\widetilde{Y}_1, \widetilde{Y}_2 \in \widetilde{V}_1$. Since $\widetilde{Y}_i = LY_i$, where $Y_i \in V_1$, $i = 1, 2$, it follows that

$$\widetilde{Y} = [\widetilde{Y}_1, \widetilde{Y}_2] = [LY_1, LY_2] = L[Y_1, Y_2] = LY,$$

where $Y \in V_2$. Thus, $\widehat{D}\varphi(x)(V_2) = \widetilde{V}_2$.
    Similar arguments show that $\widehat{D}\varphi(x)(V_i) = \widetilde{V}_i$ for all $i = 3, \ldots, M$. Thus, $\widehat{D}\varphi(x)[\mathbb{G}_x\mathbb{M}] = \mathbb{G}_{\varphi(x)}\widetilde{\mathbb{M}}$, and $\mathrm{rank}\,\widehat{D}\varphi(x) = \widetilde{N}$.
    (b) Note that the image of every basis vector field under the *hc*-differential $\widehat{D}\varphi(x)$ is a vector field of the same degree or the zero vector field (i. e., the *hc*-differential cannot decrease the degree of each basis vector field). If $\widehat{D}\varphi(x)(V_1) \subsetneq \widetilde{V}_1$ then $\widehat{D}\varphi(x)^{-1}(\widetilde{V}_1 \setminus [\widehat{D}\varphi(x)(V_1)]) = \varnothing$, and $\widehat{D}\varphi(x)[\mathbb{G}_x\mathbb{M}] \neq \mathbb{G}_{\varphi(x)}\widetilde{\mathbb{M}}$. Consequently, $\mathrm{rank}\,\widehat{D}\varphi(x) < \widetilde{N}$.
    We prove claims **(a)** and **(b)** simultaneously. The properties of the Riemannian differential $D\varphi(x)$ (see (2.4)) imply that $D\varphi(x)(H_k) \subset \widetilde{H}_k$ for $k = 1, \ldots, M$. In particular, $D\varphi(x)(H_1) \subset \widetilde{H}_1$ by the definition of a contact mapping. Moreover, $D\varphi(x)(H_1) = \widetilde{H}_1$ if and only if $\widehat{D}\varphi(x)(V_1) = \widetilde{V}_1$.

In order to verify that $\mathcal{D}\varphi(x)(H_2/H_1) \subset \widetilde{H}_2/\widetilde{H}_1$, take $\mathcal{Y} \in H_2/H_1$. Consider $\mathcal{Y}$ as a class of sums of a fixed element Y of H_2, which may have nonzero coordinates (in the basis $\{X_i\}_{i=1}^N$) only with indices greater than n_1 and less than $n_1 + n_2 + 1$, and all elements of H_1:

$$\mathcal{Y} = \{Y + T : Y \in H_2 \text{ is fixed}, T \in H_1\}.$$

Then, the structure of $D\varphi(x)$ implies that the image

$$\widetilde{\mathcal{Y}} = \mathcal{D}\varphi(x)\mathcal{Y} = D\varphi(x)Y + D\varphi(x)(H) = D\varphi(x)Y + \widetilde{H} \in \widetilde{H}_2/\widetilde{H}_1$$

consists of the vectors with only the first $\tilde{n}_1 + \tilde{n}_2$ nonzero components. Moreover, the components with indexes from $\tilde{n}_1 + 1$ to $\tilde{n}_2$ are the same for all vectors from $\widetilde{\mathcal{Y}}$. (To verify that, it suffices to write the vectors Y and T in the basis $\{X_i\}_{i=1}^N$, and the matrix of $D\varphi$ in the bases $\{X_i\}_{i=1}^N$ and $\{\widetilde{X}_i\}_{i=1}^{\widetilde{N}}$, and consider the product of this matrix and these vectors.) Thus, $\mathcal{D}\varphi(x)(H_2/H_1) \subset \widetilde{H}_2/\widetilde{H}_1$.

Verify that $\mathcal{D}\varphi(x)(H_2/H_1) \supset \widetilde{H}_2/\widetilde{H}_1$. Take $\widetilde{\mathcal{Y}} \in \widetilde{H}_2/\widetilde{H}_1$ and assume on the contrary that $\widetilde{\mathcal{Y}} \notin \mathcal{D}\varphi(x)(H_2/H_1)$. Consider a vector field $\widetilde{Y} \in \widetilde{\mathcal{Y}}$; then, $\widetilde{Y} = \widetilde{Y}_2 + \widetilde{Y}_1$ with $\widetilde{Y}_1 \in \widetilde{H}_1$. Without loss of generality we may assume that $\widetilde{Y}_1 = 0$. Since rank $\widehat{D}\varphi(x) = \widetilde{N}$, it follows that the images $D\varphi(x)(T_x\mathbb{M})$ and $(D\varphi)_{\widetilde{M}}(x)(T_x\mathbb{M})$ coincide, as in both cases we obtain the whole $T_{\varphi(x)}\widetilde{\mathbb{M}}$. Here the linear operator $(D\varphi)_{\widetilde{M}}(x)$ acts on $T_x\mathbb{M}$ (instead of $\mathbb{G}_x\mathbb{M}$), and its matrix in the bases $\{X_i(x)\}_{i=1}^N$ and $\{\widetilde{X}_i(\varphi(x))\}_{i=1}^{\widetilde{N}}$ equals that of $\widehat{D}\varphi(x)$ (written in the bases $\{\widehat{X}_i^x(x)\}_{i=1}^N$ and $\{\widehat{\widetilde{X}}_i^{\varphi(x)}(\varphi(x))\}_{i=1}^{\widetilde{N}}$). Denote by L the matrix of this operator. Then, there exists a vector Y_0 with $LY_0 = \widetilde{Y}$. Without loss of generality we may assume that Y_0 has at most n_2 nonzero components with indices from $n_1 + 1$ to $n_1 + n_2$. Consequently,

$$\widetilde{Y} = LY_0 = D\varphi(x)Y_0 + [L - D\varphi(x)]Y_0,$$

where $D\varphi(x)Y_0 \in \widetilde{H}_2$ and $[L - D\varphi(x)]Y_0 \in \widetilde{H}_1$. Since $D\varphi(x)(H_1) = \widetilde{H}_1$, there exists $Y_1 \in D\varphi(x)^{-1}([L - D\varphi(x)]Y_0) \cap H_1$. Put $Y = Y_0 + Y_1$, and then $D\varphi(x)Y = \widetilde{Y}$ and $Y \in H_2$. Thus, $\mathcal{D}\varphi(x)(H_2/H_1) = \widetilde{H}_2/\widetilde{H}_1$ and, moreover, $D\varphi(x)(H_2) = \widetilde{H}_2$.

Similarly, we can show that $\mathcal{D}\varphi(x)(H_i/H_{i-1}) = \widetilde{H}_i/\widetilde{H}_{i-1}$ and $D\varphi(x)(H_i) = \widetilde{H}_i$, $i = 3, \ldots, \widetilde{M}$. The proposition follows. $\qquad\square$

Claim **I** implies

Corollary 2.22. *The existence of $x \in \mathbb{M}$ with rank $\widehat{D}\varphi(x) = \widetilde{N}$ implies that $n_i \geq \tilde{n}_i$, $i = 1, \ldots, M$.*

Definition 2.23. The set
$$\chi = \{x \in \mathbb{M} \setminus Z : \operatorname{rank} \widehat{D}\varphi(x) < \widetilde{N}\}$$
is called the *characteristic* set. The points of χ are called *characteristic points.*

For $t \in \widetilde{\mathbb{M}}$, denote the intersection $\varphi^{-1}(t) \cap \chi$ by χ_t.

Remark 2.24. For $\widetilde{\mathbb{M}} = \mathbb{R}$ this definition coincides with the definition of the characteristic set given by P. Pansu [Pa] and J. Heinonen [He]: in this case the condition $\operatorname{rank} \widehat{D}\varphi(x) < 1$ implies that all the horizontal derivatives $X_i\varphi(x)$, $i = 1, \ldots, n$, vanish at x, and vice versa.

Remark 2.25. Proposition 2.21 explains the second part of Assumption 2.19: if $\mathbb{M} \neq \chi$ then we do not need to assume that $n_i \geq \tilde{n}_i$ for $i \geq 2$.

Assumption 2.26. If $\mathcal{H}^\nu(\chi) = \mathcal{H}^N(\chi) = 0$ then we assume that $\varphi \in C^1(\mathbb{M}, \widetilde{\mathbb{M}})$ and $X_i \in C^2(\mathbb{M})$, $i = 1, \ldots, N$ (this condition is sufficient for establishing the *hc*-differentiability of φ); otherwise, we assume that $\varphi \in C^{M+1}(\mathbb{M}, \widetilde{\mathbb{M}})$ and $X_i \in C^{M+1}(\mathbb{M})$, $i = 1, \ldots, N$.

Definition 2.27. The set
$$\mathbb{D} = \{x \in \mathbb{M} : \operatorname{rank} \widehat{D}\varphi(x) = \widetilde{N}\}$$
is called the *regular* set. If $x \in \mathbb{D}$ then we say that x is a *regular* point.

Lemma 2.28. I. *For the set*
$$\zeta = \Big\{ x \in \mathbb{M} \setminus Z : \ \exists \{X_{i_1}, \ldots, X_{i_{\widetilde{N}}}\}$$
$$\big(\operatorname{rank}([X_{i_j}\varphi](x))_{j=1}^{\widetilde{N}} = \widetilde{N}\big) \Rightarrow \Big(\sum_{j=1}^{\widetilde{N}} \deg X_{i_j} < \tilde{\nu}\Big)\Big\}$$

we have $\zeta = \varnothing$ (see Notation 2.20 for the description of Z).

II. *If there exists a family $\{X_{i_1}, \ldots, X_{i_{\widetilde{N}}}\}$ of vector fields with the properties $\operatorname{rank}([X_{i_j}\varphi](x)) = \widetilde{N}$ and $\sum\limits_{j=1}^{\widetilde{N}} \deg X_{i_j} = \tilde{\nu}$, then we have $\deg X_{i_j} \leq \widetilde{M}$, $j = 1, \ldots, \widetilde{N}$.*

Proof. Fix $x \in \mathbb{M}$ and consider the matrix of $D\varphi(x)$:

$$D\varphi(x) = \begin{pmatrix} \mathcal{V}_1(u) & * & * & * & * & * & * & \ldots & * \\ 0 & \mathcal{V}_2(u) & * & * & * & * & * & \ldots & * \\ 0 & 0 & \mathcal{V}_3(u) & * & * & * & * & \ldots & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & * & \ldots & * \\ 0 & \ldots & 0 & \vdots & \mathcal{V}_{\widetilde{M}-1}(u) & * & * & \ldots & * \\ 0 & \ldots & 0 & \ldots & 0 & \mathcal{V}_{\widetilde{M}}(u) & * & \ldots & * \end{pmatrix}.$$

Choose $\widetilde{N}$ linearly independent columns with the minimal possible sum of the corresponding degrees (we consider $\deg X_j$ as the degree of column j, $j = 1, \ldots, N$).

To this end, we must choose the maximal possible quantity of vectors from the blocks corresponding to the minimal degrees. In the first block, we can choose at most $\tilde{n}_1$ linearly independent vectors. Next, take columns from $n_1 + 1$ to $n_1 + n_2$ and the corresponding "diagonal" block. In this block, we can choose at most $\tilde{n}_2$ linearly independent $\tilde{n}_2$-dimensional elements.

On assuming that there are more than $\tilde{n}_2$ linearly independent columns, we obtain a contradiction. Indeed, since the "diagonal" block is an $(\tilde{n}_2 \times n_2)$-matrix, there exists an elementary transformation reducing at least $n_2 - \tilde{n}_2$ of its columns to zero. Apply this transformation to the columns of the matrix of $D\varphi(x)$. Then, this block of size $(\tilde{n}_1 + \tilde{n}_2) \times (n_1 + n_2)$ includes $n_1 + n_2 - \tilde{n}_2$ columns of dimension $\tilde{n}_1$ (more exactly, these column vectors belong to $\mathbb{R}^{\tilde{n}_1} \times 0^{\tilde{n}_2}$). Recall that the maximal number of linearly independent columns is $\tilde{n}_1$, and we have already chosen them in the first "diagonal" block. Suppose that there are less than $\tilde{n}_1 + \tilde{n}_2$ linearly independent columns. Then, since $\operatorname{rank} D\varphi(x) = \widetilde{N}$, a "missing" column can be "replaced" by a column of a higher degree. Thus, we can obtain the minimal possible sum of degrees if we have only $n_1 - \tilde{n}_1 + n_2 - \tilde{n}_2$ linearly dependent among the first $n_1 + n_2$ columns. Therefore, the corresponding sum of degrees equals $\tilde{n}_1 + 2\tilde{n}_2$.

Applying further the same arguments to the degrees $3, \ldots, \widetilde{M}$, we conclude that the minimal possible sum of degrees of linearly independent vector fields $\{X_{i_j}\varphi\}_{j=1}^{\widetilde{N}}$ is equal to $\tilde{\nu}$. Thus, claim **I** is proved.

Claim **II** follows since, firstly, we can obtain the sum equal to $\tilde{\nu}$ only by considering the first $\widetilde{M}$ blocks, and, secondly, if we have less than $\tilde{n}_k$ linearly independent vector fields on step $k \leq \widetilde{M}$ then the sum of degrees corresponding to the resulting collection is strictly greater than $\tilde{\nu}$. $\qquad\square$

Theorem 2.29. I. *The characteristic set χ coincides with*

$$\Big\{ x \in \mathbb{M} \setminus Z : \ \forall \{X_{i_1}, \ldots, X_{i_{\widetilde{N}}}\}$$

$$\big(\operatorname{rank}([X_{i_j}\varphi](x))_{j=1}^{\widetilde{N}} = \widetilde{N}\big) \Rightarrow \Big(\sum_{j=1}^{\widetilde{N}} \deg X_{i_j} > \tilde{\nu}\Big) \Big\}. \quad (2.6)$$

II. *The regular set $\mathbb{D}$ coincides with*

$$\Big\{ x \in \mathbb{M} \setminus Z : \ \exists \{X_{i_1}, \ldots, X_{i_{\widetilde{N}}}\}$$

$$\big(\operatorname{rank}([X_{i_j}\varphi](x))_{j=1}^{\widetilde{N}} = \widetilde{N}\big) \Rightarrow \Big(\sum_{j=1}^{\widetilde{N}} \deg X_{i_j} = \tilde{\nu}\Big) \Big\}. \quad (2.7)$$

Proof. **I. (a)** Denote the set in (2.6) by A, and verify that $A \subset \chi$. Consider a point $x \in A$. By (2.6), every $\widetilde{N}$ columns of the matrix of $D\varphi(x)$ with indices $i_1, \ldots, i_{\widetilde{N}}$ corresponding to a collection $\{X_{i_1}, \ldots, X_{i_{\widetilde{N}}}\}$ with $\sum_{j=1}^{\widetilde{N}} \deg X_{i_j} = \tilde{\nu}$ are linearly dependent. Assume on the contrary that the rank of the matrix of $\widehat{D}\varphi(x)$ equals $\widetilde{N}$, and consequently, there exist $\widetilde{N}$ linearly independent columns in the matrix of $\widehat{D}\varphi(x)$. The matrix of $\widehat{D}\varphi(x)$ has a block structure, where the block k is an $(\tilde{n}_k \times n_k)$-matrix. Thus, in each block, only $\tilde{n}_k$ columns can be linearly independent. Consequently, the sum of the degrees of the vector fields corresponding to these linearly independent columns equals $\tilde{\nu}$.

The relation between the matrices of $D\varphi(x)$ and $\widehat{D}\varphi(x)$ implies that the corresponding columns of the matrix of $D\varphi(x)$ are also linearly independent, and the sum of the degrees of the corresponding vector fields is equal to $\tilde{\nu}$. Thus, we arrive at a contradiction.

The argument above implies that $A \subset \chi$.

(b) Verify that $\chi \subset A$. Consider $x \in \chi$. Since rank $\widehat{D}\varphi(x) < \widetilde{N}$, it follows that every $\widetilde{N}$ columns are linearly dependent. Our goal is to show that if we take the columns of the matrix of $D\varphi(x)$ with indices $i_1, \ldots, i_{\widetilde{N}}$ such that

$$\sum_{j=1}^{\widetilde{N}} \deg X_{i_j} \leq \tilde{\nu}$$

then

$$\operatorname{rank}([X_{i_j}\varphi](x))_{j=1}^{\widetilde{N}} < \widetilde{N}.$$

In view of claim **I** of Lemma 2.28, it suffices to consider the columns with indices $i_1, \ldots, i_{\widetilde{N}}$ satisfying

$$\sum_{j=1}^{\widetilde{N}} \deg X_{i_j} = \tilde{\nu}.$$

Take $\widetilde{N}$ columns $c_{i_1}, \ldots, c_{i_{\widetilde{N}}}$ of the matrix of $\widehat{D}\varphi(x)$ corresponding to some vectors $X_{i_1}, \ldots, X_{i_{\widetilde{N}}}$ with $\sum_{j=1}^{\widetilde{N}} \deg X_{i_j} = \tilde{\nu}$. Since

$$\operatorname{rank}(\{c_{i_1}, \ldots, c_{i_{\widetilde{N}}}\}) < \widetilde{N},$$

there exists a transformation τ of the matrix of $\widehat{D}\varphi(x)$ taking at least one of c_{i_j} to a zero column, and preserving the block structure of the initial matrix. Denote this column by $c_{i_{j_0}}$. Here we assume that j_0 is the minimal

number with this property. Put $k = \deg X_{i_{j_0}} - 1$. Because of the block structure of the matrix $\widehat{D}\varphi(x)$, the numbers of nonzero entries of $c_{i_{j_0}}$ are at least $\dim \widetilde{H}_k + 1$, and at most $\dim \widetilde{H}_{k+1}$. Without loss of generality we may assume that $i_{j_0} = \dim H_k + 1$. Since we cannot transform all preceding columns into zero columns, it follows that $j_0 = \dim \widetilde{H}_k + 1$.

Apply the same transformation τ to the matrix of $D\varphi(x)$, and consider the images of j_0 columns with indices $i_1, \ldots, i_{j_0}$. Note that we may regard the image of column $i_{j_0} = \dim H_k + 1$ as an element of $\mathbb{R}^{\dim \widetilde{H}_k}$. Taking the structure of $D\varphi(x)$ into account, we have $j_0 = \dim \widetilde{H}_k + 1$ vectors (columns) belonging to the space $\mathbb{R}^{\dim \widetilde{H}_k}$ (because the components with indices greater than $\dim \widetilde{H}_k$ vanish). Thus, the rank of this collection equals $\dim \widetilde{H}_k < j_0$. Consequently, the rank of $\{X_{i_1}\varphi, \ldots, X_{i_{\widetilde{N}}}\varphi\}$ is strictly less than $\widetilde{N}$. Since the collections of $\widetilde{N}$ vectors with the sum of degrees equal to $\tilde{\nu}$ appear only in the matrix of $\widehat{D}\varphi(x)$ (see Lemma 2.28), we see that if the sum of the degrees is equal to $\tilde{\nu}$, then the rank is strictly less than $\widetilde{N}$. By taking Lemma 2.28 and the fact that $\mathrm{rank}(D\varphi(x)) = \widetilde{N}$ into account, we see that $\chi \subset A$.

II. (a) Denote the set in (2.7) by B. Assume the contrary and take $x \in B$ with $\mathrm{rank}(\widehat{D}\varphi(x)) < \widetilde{N}$. Step **I** yields $x \in A$, and thus, we obtain a contradiction since $A \cap B = \varnothing$. Consequently, $B \subset \mathbb{D}$.

(b) Take $x \in \mathbb{D}$. Since $\mathrm{rank}\,\widehat{D}\varphi(x) = \widetilde{N}$, it follows that $\mathrm{rank}\,D\varphi(x) = \widetilde{N}$ and $x \notin Z$. Take an arbitrary collection of $\widetilde{N}$ linearly independent columns of the matrix of $\widehat{D}\varphi(x)$. Consequently, the corresponding columns of the matrix of $D\varphi(x)$ are linearly independent as well, and $x \in B$. Thus, $\mathbb{D} \subset B$.

The theorem follows. $\qquad\square$

3   Properties of Level Sets

In this section, we assume that $x \in \mathbb{M} \setminus Z$.

First of all, we introduce a new metric, which is equivalent to the initial one and simplifies our computations.

Definition 3.1. Let $\mathbb{M}$ be a Carnot manifold of topological dimension N and depth M, and put $x = \exp\Big(\sum\limits_{i=1}^{N} x_i X_i\Big)(g)$. Define the distance $d_2(x, g)$ as

follows:

$$d_2(x, g) = \max\Big\{ \Big(\sum_{j=1}^{n_1} |x_j|^2\Big)^{\frac{1}{2}},$$

$$\Big(\sum_{j=n_1+1}^{n_1+n_2} |x_j|^2\Big)^{\frac{1}{2 \cdot \deg X_{n_1+1}}}, \dots, \Big(\sum_{j=N-n_M+1}^{N} |x_j|^2\Big)^{\frac{1}{2 \cdot \deg X_N}}\Big\}.$$

A similar metric d_2^u is introduced on the local Carnot group $\mathcal{G}^u\mathbb{M}$.

The set $\{y \in \mathbb{M} : d_2(y, x) < r\}$ is called the ball of radius $r > 0$ centered at x and denoted by $\mathrm{Box}_2(x, r)$. Similarly, $\mathrm{Box}_2^u(x, r)$ stands for the ball in d_2^u of radius $r > 0$ centered at x.

Remark 3.2. The preimage of $\mathrm{Box}_2(x, r)$ in the metric d_2 under the mapping θ_x equals

$$\mathrm{Box}_2(0, r) = B_2^{n_1}(x, r) \times B_2^{n_2}(x, r^2) \times \dots \times B_2^{n_M}(x, r^M),$$

where $B_2^{n_i}$ is a Euclidean ball of dimension n_i, $i = 1, \dots, M$.

Observe that $\mathrm{Box}_2^u(u, r) = \mathrm{Box}_2(u, r)$ in the quasimetric d_2^u since

$$\exp\Big(\sum_{i=1}^{N} x_i \widehat{X}_i^u\Big)(u) = \exp\Big(\sum_{i=1}^{N} x_i X_i\Big)(u) \qquad (3.1)$$

for all collections $\{x_i\}_{i=1}^{N}$ such that both parts of (3.1) make sense [Vod4].

The following proposition is useful for proving the main results.

Proposition 3.3. *Let $\mathbb{M}$ be a Carnot manifold of topological dimension N and depth M. Given a sufficiently small compact domain $U \Subset \mathbb{M}$, there exist positive constants $C > 0$ and $r_0 > 0$ depending on U, M, and N such that all points u and v of U satisfy*

$$\bigcup_{x \in \mathrm{Box}_2^u(v, r)} \mathrm{Box}_2^u(x, \xi) \subseteq \mathrm{Box}_2^u(v, r + C\xi), \quad 0 < \xi,\, r \le r_0.$$

Proof. The proof follows the scheme of proof of the similar lemma for boxes in the metric d_∞^u of [VodKm1, KmVod].

Put $x = \exp\Big(\sum_{i=1}^{N} x_i \widehat{X}_i^u\Big)(v)$, $d_2^u(v, x) \le r$, and $z = \exp\Big(\sum_{i=1}^{N} z_i \widehat{X}_i^u\Big)(x)$, $d_2^u(x, z) \le \xi$. Estimate the distance $d_2^u(v, z)$ applying group operation to points x and z. Namely, estimate the coefficients $\{\zeta_i\}_{i=1}^{N}$ satisfying $z = \exp\Big(\sum_{i=1}^{N} \zeta_i \widehat{X}_i^u\Big)(v)$.

19

CASE OF $\deg X_i = 1$. We have

$$\sum_{i=1}^{n}(\zeta_i)^2 = \sum_{i=1}^{n}(x_i + z_i)^2 = \sum_{i=1}^{n}(x_i)^2 + \sum_{i=1}^{n}(z_i)^2 + 2\sum_{i=1}^{n}(x_i z_i) \le (r + \xi)^{2\deg X_i}.$$

CASE OF $\deg X_i = 2$. We have

$$\sum_{i=n+1}^{n+n_2}(\zeta_i)^2 \le \sum_{i=n+1}^{n+n_2}\Big(x_i + z_i + \sum_{\substack{|e_l+e_j|_h=2,\\ l<j}} \widehat{F}^i_{e_l,e_j}(u)(x_l z_j - z_l x_j)\Big)^2$$

$$\le r^4 + \xi^4 + 2r^2\xi^2 + \sum_{i=n+1}^{n+n_2}\Big(\sum_{\substack{|e_l+e_j|_h=2,\\ l<j}} \widehat{F}^i_{e_l,e_j}(u)(x_l z_j - z_l x_j)\Big)^2$$

$$+ 2\sum_{i=n+1}^{n+n_2}\Big((x_i + z_i)\sum_{\substack{|e_l+e_j|_h=2,\\ l<j}} \widehat{F}^i_{e_l,e_j}(u)(x_l z_j - z_l x_j)\Big)$$

$$\le r^4 + \xi^4 + c_i(u)r^2\xi^2 + b_i(u)r\xi(r + \xi)^2$$

$$\le (r + a_2(u)\xi)^4 = (r + a_2(u)\xi)^{2\deg X_i},$$

where $b_i(u)$ and $c_i(u)$ are linear combinations of the functions $\{\widehat{F}^i_{e_l,e_j}(u)\}_{l,j}$ on assuming that $|x_l z_j - z_l x_j| = 2r\xi$ and $|x_i + z_i| = r + \xi$ for all i, l, j. They are continuous with respect to $u \in U$. We can represent each sum $r^4 + \xi^4 + c_i(u)r^2\xi^2 + b_i(u)r\xi(r + \xi)^2$ as $(r + d_i(u)\xi)^4$, where $d_i(u)$ depends on $b_i(u)$ and $c_i(u)$. Put $a_2(u) = \max_{i:\deg X_i=2} d_i(u)$ and assume without loss of generality that $a_2(u) \ge 1$.

CASE OF $\deg X_i = k > 2$. Denote the sum $n + \sum_{j=2}^{k} n_j$ by S_k. Then, as in the previous case of $\deg X_i = 2$, we obtain

$$\sum_{i=S_{k-1}+1}^{S_k}(\zeta_i)^2 \le \sum_{i=S_{k-1}+1}^{S_k}\Big(x_i + z_i + \sum_{\substack{|\mu+\beta|_h=k,\\ \mu>0,\beta>0}} |\widehat{F}^i_{\mu,\beta}(u)|x^\mu \cdot z^\beta\Big)^2$$

$$\le (r + a_k(u)\xi)^{2k} = (r + a_k(u)\xi)^{2\deg X_i}.$$

Here we use the property

$$\sum_{\substack{|\mu+\beta|_h=k,\\ \mu>0,\beta>0}} |\widehat{F}^i_{\mu,\beta}(u)|x^\mu \cdot z^\beta \le \sum_{\substack{|\mu+\beta|_h=k,\\ \mu>0,\beta>0}} c^i_{\mu,\beta}(u)r^{|\mu|_h} \cdot \xi^{|\beta|_h},$$

and define each function $a_k(u)$ in the similar way as $a_2(u)$ in the case $\deg X_i = 2$. We also assume without loss of generality that $a_k(u), c^i_{\mu,\beta}(u) \geq 1$. Put $a(u) = \max_i a_i(u)$. The estimates above yield

$$d_2^u(v, x) = \max\Big\{ \Big(\sum_{j=1}^{n} |\zeta_j|^2\Big)^{\frac{1}{2}},$$

$$\Big(\sum_{j=n+1}^{n+n_2} |\zeta_j|^2\Big)^{\frac{1}{2\cdot\deg X_{n+1}}}, \ldots, \Big(\sum_{j=N-n_M+1}^{N} |\zeta_j|^2\Big)^{\frac{1}{2\cdot\deg X_N}}\Big\}$$

$$\leq \max_i\{(r + a_i(u)\xi)^{\frac{\deg X_i}{\deg X_i}}\} \leq r + a(u)\xi.$$

Since all $a_i(u)$ are continuous with respect to u, we may choose sufficiently large $C < \infty$ with $a(u) \leq C$ for all u belonging to the given compact domain $U \Subset \mathbb{M}$. The lemma follows. $\square$

To prove the main theorems, we need a convenient quasimetric equivalent to the Riemannian metric.

Definition 3.4. Given

$$y = \exp\Big(\sum_{i=1}^{N} y_i X_i\Big)(x),$$

put $\rho(y, x) = \max_{i=1,\ldots,N}\{|y_i|\}$.

Notation 3.5. In Theorems 3.7 and 3.11, we establish some local results for a fixed point x. From now on we use the auxiliary mapping $\psi = \varphi \circ \theta_x$.

Notation 3.6. Put

$$\nu_0(x) = \min\Big\{\nu : \exists\{X_{i_1}, \ldots, X_{i_{\widetilde{N}}}\}$$

$$\big(\mathrm{rank}([X_{i_j}\varphi](x))_{j=1}^{\widetilde{N}} = \widetilde{N}\big) \Rightarrow \Big(\sum_{j=1}^{\widetilde{N}} \deg X_{i_j} = \nu\Big)\Big\}.$$

It is clear that $\nu_0|_\chi > \tilde{\nu}$ and $\nu_0|_\mathbb{D} = \tilde{\nu}$.

Theorem 3.7. *Fix $x \in \varphi^{-1}(t)$. Then, in a neighborhood in $\mathbb{R}^N$, the $\mathcal{H}^{N-\widetilde{N}}$-measure of $T_0[\psi^{-1}(t)] \cap \mathrm{Box}_2(0, r)$ (see Remark 3.2) is equal to*

$$Cr^{\nu-\nu_0(x)}(1 + o(1)) \tag{3.2}$$

where C is independent of r, and $o(1) \to 0$ as $r \to 0$.

Remark 3.8. We emphasize that the mapping $\psi = \varphi \circ \theta_x$ acts on a neighborhood of the origin in $\mathbb{R}^N$. Consequently, the tangent plane to the level set $\psi^{-1}(t)$ lies in $\mathbb{R}^N$, and the intersection $T_0[\psi^{-1}(t)] \cap \mathrm{Box}_2(0, r)$ is well-defined.

Proof of Theorem 3.7. We split the proof into 6 steps. On step **I** we choose a suitable basis $\{w_j\}_{j=1}^N$ for the tangent space $\mathcal{T} = T_0[\psi^{-1}(t)]$ to the level set. On step **II** we define two projections of the basis vectors in $T_0[\psi^{-1}(t)]$. In particular, the first projection π assigns to each basis vector w_j some vector p_j of the same degree, and the second projection assigns a standard vector $\frac{\pi_j}{|\pi_j|}$ in $\{e_1, \ldots, e_N\}$ to each basis vector w_j for $T_0[\psi^{-1}(t)]$. On step **III** we show that $\mathrm{rank}(\sigma_i \psi)_{i=1}^{\widetilde{N}} = \widetilde{N}$ (henceforth $\sigma_i \psi$ stands for the action of the vector σ_i on ψ, $i = 1, \ldots, N$), where

$$\{\sigma_1, \ldots, \sigma_{\widetilde{N}}\} = \{e_1, \ldots, e_N\} \setminus \left\{ \frac{\pi_1}{|\pi_1|}, \ldots, \frac{\pi_{N-\widetilde{N}}}{|\pi_{N-\widetilde{N}}|} \right\},$$

and on step **IV** we prove that the sum of the degrees of σ_i, $i = 1, \ldots, \widetilde{N}$, coincides with $\nu_0(x)$. Consequently, the sum of the degrees of π_j, $j = 1, \ldots, N - \widetilde{N}$, equals $\nu - \nu_0(x)$. Further, on step **V** we deduce that the Lebesgue measure of $\mathrm{Box}_2(0, r) \cap \mathrm{span}\{p_1, \ldots, p_{N-\widetilde{N}}\}$ equals $Cr^{\nu - \nu_0(x)}$, where C is independent of r. Finally, on step **VI** we prove that the length of $\mathbb{R} w_j \cap \mathrm{Box}_2(0, r)$ equals $O(r^{k(j)})$ for sufficiently small $r > 0$, and applying this result we show that $\pi(\mathcal{T} \cap \mathrm{Box}_2(0, r))$ coincides with the $o(r)$-neighborhood of $\mathcal{S} \cap \mathrm{Box}_2(0, r)$ in $\mathcal{S}$, where $\mathcal{S} = \mathrm{span}\{p_1, \ldots, p_{N-\widetilde{N}}\}$ and $o(r)$ is taken with respect to the metric d_2. The theorem follows from the last result.

Step I. Consider the normal coordinates at the point x. Recall that $\mathcal{T} = T_0[\psi^{-1}(t)]$. Choose an arbitrary basis in $\mathcal{T}$, and write it as a matrix in which the basis vectors are written as rows.

(i) By elementary row transformations, reduce this matrix to

$$\Delta = \begin{pmatrix} * & \ldots & * & 0 & & \ldots & & & & 0 \\ * & \ldots & * & * & 0 & \ldots & 0 & & \ldots & 0 \\ * & & \ldots & & * & * & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & & & & \ddots & \ddots & & 0 \\ * & & \ldots & & & & & * & 0 & 0 \\ * & & \ldots & & & & & * & * & * \end{pmatrix},$$

where the upper right triangle consists of zeroes, and the last nonzero entries of the rows appear in distinct columns.

(ii) In the matrix Δ there is the following natural "grading" of the columns corresponding to the grading of the Lie algebra V of vector fields

on the local tangent cone. Split the columns of Δ into M blocks $\mathcal{B}_1, \ldots, \mathcal{B}_M$ such that $\mathcal{B}_k$ consists of columns from $\dim H_{k-1} + 1$ to $\dim H_k$:

$$
\begin{pmatrix}
{}^{1} \quad {}^{\cdots} \quad {}^{\dim H_1} & {}^{\dim H_1+1} \quad {}^{\cdots} \quad {}^{\dim H_2} & {}^{\cdots\cdots\cdots} & {}^{\dim H_{M-1}+1} \quad {}^{\cdots} \quad {}^{N} \\
\mathcal{B}_1 & \mathcal{B}_2 & \cdots\cdots & \mathcal{B}_M
\end{pmatrix}.
$$

Next, there is also a "grading" of rows of Δ. There are M blocks $\mathcal{A}_1, \ldots, \mathcal{A}_M$. Here $\mathcal{A}_l$ consists of the rows whose last nonzero element appears in a column with index in $[\dim H_{l-1} + 1, \dim H_l]$:

$$
\begin{pmatrix}
\mathcal{A}_1 \\
\mathcal{A}_2 \\
\vdots \\
\mathcal{A}_M
\end{pmatrix}
=
\left(
\begin{array}{ccccccccc}
{}^{1} & {}^{\cdots} & {}^{\dim H_1} & {}^{\dim H_1+1} & {}^{\cdots} & {}^{\dim H_2} & {}^{\cdots} & {}^{\dim H_{M-1}+1} & {}^{\cdots} \quad {}^{N} \\
 & \cdots & * & 0 & & & \cdots & & 0 \\
* & \cdots & * & * & 0 & \cdots & 0 & & \cdots \quad 0 \\
* & & \cdots & & * & * & 0 & 0 & \cdots \quad 0 \\
\vdots & & \vdots & & & & \ddots & \ddots & 0 \\
* & & \cdots & & & & & * & 0 \quad 0 \\
* & & \cdots & & & & & * & * \quad *
\end{array}
\right).
$$

Some of these blocks can be empty.

(iii) For each $k = 1, \ldots, M$, put $\mathcal{V}_k = \mathcal{B}_k \cap \mathcal{A}_k$

$$
\begin{pmatrix}
 & \mathcal{V}_1 & 0 & 0 & & \cdots & & 0 \\
* & * & & \mathcal{V}_2 & 0 & \cdots & 0 & & \cdots & 0 \\
* & & * & & * & 0 & 0 & & \cdots & 0 \\
\vdots & \mathcal{B}_1 \setminus \mathcal{V}_1 & \vdots & \mathcal{B}_2 \setminus \mathcal{V}_2 & * & \ddots & \ddots & & 0 \\
* & & & & \vdots & \vdots & * & \mathcal{V}_{M-1} & 0 \\
* & & * & & * & * & & \mathcal{B}_{M-1} \setminus \mathcal{V}_{M-1} & \mathcal{V}_M
\end{pmatrix}.
$$

Further, we transform the blocks $\mathcal{B}_k \setminus \mathcal{V}_k$, $k = 1, \ldots, M$. Define the transformation of Δ by induction. For $k = M$, we have nothing to transform; thus, the base of induction holds.

Suppose that we have transformed the blocks $\mathcal{B}_k \setminus \mathcal{V}_k$, $k = l + 1, \ldots, M$, $l \leq M - 1$, and assume that $\mathcal{V}_l \neq \varnothing$ (otherwise, we have nothing to transform). Replace the blocks $\mathcal{A}_{l+1}, \ldots, \mathcal{A}_M$ by the projections of their row vectors onto $(\mathrm{span}\{\mathcal{A}_l\})^{\perp} \cap \mathrm{span}\{\mathcal{A}_l, \ldots, \mathcal{A}_M\}$. Roughly speaking, we remove the part collinear to $\mathcal{A}_l$ from the row vectors of $\mathcal{A}_{l+1}, \ldots, \mathcal{A}_M$. This projection preserves the blocks $\mathcal{B}_k \setminus \mathcal{V}_k$, $k = l + 1, \ldots, M$, $l \leq M - 1$, because of the "triangular" structure of Δ.

Moreover, the rows of $\mathcal{B}_k \setminus \mathcal{V}_k$ are orthogonal to $\mathcal{V}_k$ with respect to the classical inner product.

Denote the resulting vectors by $w_1, \ldots, w_{N-\widetilde{N}}$:

$$W = \begin{pmatrix} \mathcal{V}_1 & 0 & 0 & & \ldots & & & 0 \\ * & * & \mathcal{V}_2 & 0 & \ldots & 0 & & \ldots & 0 \\ * & & * & * & & 0 & 0 & \ldots & 0 \\ \vdots & \mathcal{V}_1^\perp & \vdots & \mathcal{V}_2^\perp & * & & \ddots & \ddots & & 0 \\ * & & & & \vdots & & \vdots & * & \mathcal{V}_{M-1} & 0 \\ * & & * & & * & & * & & \mathcal{V}_{M-1}^\perp & \mathcal{V}_M \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ \vdots \\ w_{N-\widetilde{N}} \end{pmatrix}.$$
(3.3)

Thus, we have constructed a "suitable" basis for $\mathcal{T}$.

Step II. For each $j = 1, \ldots, N - \widetilde{N}$, define the number $k(j)$ as follows. Let $l(j)$ be the row index of the last nonzero element of w_j in (3.3), and put $k(j) = \deg X_{l(j)}$.

Project each w_j onto $\mathcal{V}_{k(j)}$ by taking the last nonzero coordinates belonging to $\mathcal{V}_{k(j)}$, and denote this projection by $p_j = \pi_{\mathcal{T}}(w_j)$.

By the properties of W, this projection is orthogonal with respect to the standard Riemannian metric on $\mathbb{R}^N$: for each vector $w_j \in \mathcal{T}$ we have $(w_j - \pi_{\mathcal{T}}(w_j)) \perp \pi_{\mathcal{T}}(\mathcal{T})$. Indeed, expand

$$w_j = p_j + q_j = \pi_{\mathcal{T}}(w_j) + q_j$$

and verify that q_j is orthogonal to $\pi_{\mathcal{T}}(\mathcal{T})$. Take $u \in \pi_{\mathcal{T}}(\mathcal{T})$. First, suppose that u is a "basis" vector in this image: u is one of the row-vectors of the block $\mathcal{V}_k$ for some $1 \le k \le M$. If $p_j \in \mathcal{V}_l$ and $l > k$ then $q_j \perp u$ by choice of W. If $l \le k$ then obviously $q_j \perp u$ since q_j and u have no nonzero coordinates with the same indices. Since j is an arbitrary number and u is an arbitrary basis vector, it follows that all q_j are orthogonal to $\pi_{\mathcal{T}}(\mathcal{T})$. Consequently, $\text{span}\{q_j, j = 1, \ldots, N - \widetilde{N}\} \perp \pi_{\mathcal{T}}(\mathcal{T})$.

Denote by π_j the vector obtained from w_j by putting

$$\begin{cases} (\pi_j)_{l(j)} & = (w_j)_{l(j)}, \\ (\pi_j)_k & = 0 \text{ for } k \ne l(j). \end{cases}$$

Put $\deg \pi_j = k(j)$. By construction, the sum of the degrees of $\pi_1, \ldots, \pi_{N-\widetilde{N}}$ equals the sum of degrees of the tangent vectors $(\theta_x^{-1})_* \langle w_1 \rangle, \ldots, (\theta_x^{-1})_* \langle w_{N-\widetilde{N}} \rangle$.

Step III. Consider the vectors

$$\{\sigma_1, \ldots, \sigma_{\widetilde{N}}\} = \{e_1, \ldots, e_N\} \setminus \left\{ \frac{\pi_1}{|\pi_1|}, \ldots, \frac{\pi_{N-\widetilde{N}}}{|\pi_{N-\widetilde{N}}|} \right\}$$
(3.4)

of the standard basis in $\mathbb{R}^N$. Since θ_x is a diffeomorphism, we have

$$\text{rank}(([(\theta_x)_*(0)\langle \sigma_i \rangle] \varphi)(x))_{i=1}^{\widetilde{N}} = \widetilde{N} \iff \text{rank}([\sigma_i \psi](0))_{i=1}^{\widetilde{N}} = \widetilde{N}.$$

Here the symbol $(\theta_x)_*(0)\langle\sigma_i\rangle = D\theta_x(0)\langle\sigma_i\rangle$ stands for the action of the differential of θ_x at 0 on the vector σ_i, $i = 1, \ldots, \widetilde{N}$.

Verify that the rank of $([\sigma_i\psi](0))_{i=1}^{\widetilde{N}}$ equals $\widetilde{N}$. Indeed, assume the contrary. Then $\ker D\psi(0) \cap \mathrm{span}\{\sigma_1, \ldots, \sigma_{\widetilde{N}}\} \neq 0$, and therefore

$$\dim \mathcal{T} \cap \mathrm{span}\{\sigma_1, \ldots, \sigma_{\widetilde{N}}\} \geq 1.$$

Consider the coordinates of $w \in \mathcal{T} \cap \mathrm{span}\{\sigma_1, \ldots, \sigma_{\widetilde{N}}\}$. On the one hand, since $w \in \mathcal{T}$, the choice of the matrix W implies that we can expand w as

$$w = \sum_{i=1}^{N-\widetilde{N}} a_i w_i.$$

Put $i_0 = \max\{i : a_i \neq 0\}$. Since $w_i = (w_i - \pi_i) + \pi_i$, it follows that

$$w = \sum_{i=1}^{N-\widetilde{N}} a_i(w_i - \pi_i) + a_i\pi_i = \Big(\sum_{i=1}^{i_0-1} a_i[(w_i - \pi_i) + \pi_i] + a_{i_0}(w_{i_0} - \pi_{i_0})\Big) + a_{i_0}\pi_{i_0}.$$

Recall that by the choice of $\{\pi_i\}_{i=1}^{N-\widetilde{N}}$, the only nonzero coordinate of π_{i_0} has index $l(i_0)$, and the nonzero coordinates of

$$\Big(\sum_{i=1}^{i_0-1} a_i[(w_i - \pi_i) + \pi_i] + a_{i_0}(w_{i_0} - \pi_{i_0})\Big)$$

have indices strictly less than $l(i_0)$. On the other hand, since $w \in \mathrm{span}\{\sigma_1, \ldots, \sigma_{\widetilde{N}}\}$ it follows from (3.4) that all coefficients of $\pi_1, \ldots, \pi_{N-\widetilde{N}}$ in this expansion must vanish. Consequently, $a_{i_0} = 0$. Thus, we arrive at a contradiction.

Step IV. Verify that the sum of degrees of basis vector fields $\{X_{i_j}\}_{j=1}^{\widetilde{N}}$ such that $X_{i_j}(x) = (\theta_x)_*(0)\langle\sigma_j\rangle$, $j = 1, \ldots, \widetilde{N}$, equals $\nu_0(x)$.

(i) Assume on the contrary that there exist standard vectors $\delta_1, \ldots, \delta_{\widetilde{N}}$ with

$$\nu_0(x) = \sum_{i=1}^{\widetilde{N}} \deg \delta_i < \sum_{i=1}^{\widetilde{N}} \deg \sigma_i \tag{3.5}$$

and $\mathrm{rank}(\delta_i\psi)_{i=1}^{\widetilde{N}} = \widetilde{N}$ (here $\deg \delta_i$ stands for the value $\deg[(\theta_x)_*(0)\langle\delta_i\rangle]$, $i = 1, \ldots, \widetilde{N}$). Since $\{\delta_1, \ldots, \delta_{\widetilde{N}}\} \neq \{\sigma_1, \ldots, \sigma_{\widetilde{N}}\}$, there exists at least one $\frac{\pi_j}{|\pi_j|} = \delta_k$ for some j and k. Consequently, there exists a vector v satisfying $v + \frac{\pi_j}{|\pi_j|} \in \mathcal{T}$. Moreover, the index of the last nonzero component of v is strictly less than $l(j)$ (see the matrix W of (3.3)).

25

(ii) Consider the chosen vector v. It is evident that $v \notin \mathcal{T}$ (otherwise, $\pi_j \in \mathcal{T}$ and $\delta_k \in \mathcal{T}$, which is impossible). Observe that $\mathrm{rank}(\tau_i \psi)_{i=1}^{\widetilde{N}} < \widetilde{N}$, where $\tau_k = \frac{1}{2}\left(v + \frac{\pi_j}{|\pi_j|}\right) \in \mathcal{T}$, and $\tau_i = \delta_i$ for all $i \neq k$. Consequently, $\mathrm{rank}(\lambda_i \psi)_{i=1}^{\widetilde{N}} = \widetilde{N}$, where $\lambda_k = v$ and $\lambda_i = \delta_i$ for all $i \neq k$. Indeed, assume on the contrary that $\mathrm{rank}(\lambda_i \psi)_{i=1}^{\widetilde{N}} < \widetilde{N}$. Then

$$(\delta_i \psi)_{i=1}^{\widetilde{N}} = 2(\tau_i \psi)_{i=1}^{\widetilde{N}} - (\lambda_i \psi)_{i=1}^{\widetilde{N}}. \qquad (3.6)$$

The assumption implies that $v\psi = \sum_{i=1}^{\widetilde{N}} a_i [\delta_i \psi]$ because $\mathrm{rank}(\delta_i \psi)_{i=1}^{\widetilde{N}} = \widetilde{N}$. Since $2\tau_k \psi = \left(v + \frac{\pi_j}{|\pi_j|}\right)\psi = 0$, it follows from (3.6) that the column k of $(\delta_i \psi)_{i=1}^{\widetilde{N}}$ is equal to $-v\psi = -\sum_{i=1}^{\widetilde{N}} a_i[\delta_i \psi]$, and, consequently, $\mathrm{rank}(\delta_i \psi)_{i=1}^{\widetilde{N}} < \widetilde{N}$. Thus, we arrive at a contradiction.

Since $\mathrm{rank}(\lambda_i \psi)_{i=1}^{\widetilde{N}} = \widetilde{N}$, assuming the contrary and applying similar arguments, we infer that there exists at least one coordinate $(v)_j$ of v such that $\mathrm{rank}(\mu_i \psi)_{i=1}^{\widetilde{N}} = \widetilde{N}$, where $\mu_k = e_j$ and $\mu_i = \delta_i$ for all $i \neq k$.

(iii) Put

$$l = \min\{j : \mathrm{rank}(\mu_i \psi)_{i=1}^{\widetilde{N}} = \widetilde{N}, \text{ where } \mu_k = e_j, \ \mu_i = \delta_i \text{ for } i \neq k\}.$$

Note that l is strictly less than $l(j)$ (the index of the only nonzero coordinate of π_j). If $\deg e_l < \deg \pi_j$ then we obtain a contradiction with the assumption that $\sum_{i=1}^{\widetilde{N}} \deg \delta_i = \nu_0(x)$. If $\deg e_l = \deg \pi_j$ then we consider the collection $\{\mu_i\}_{i=1}^{\widetilde{N}}$ instead of $\{\delta_i\}_{i=1}^{\widetilde{N}}$, and repeat the previous arguments.

(iv) Observe that after finitely many iterations of the arguments described in substeps **(ii)**–**(iii)**, we obtain a collection satisfying

$$\sum_{i=1}^{\widetilde{N}} \deg \mu_i < \sum_{i=1}^{\widetilde{N}} \deg \delta_i = \nu_0(x)$$

(here $\deg \mu_i$ stands for the value $\deg[(\theta_x)_*(0)\langle \mu_i \rangle]$, $i = 1, \ldots, \widetilde{N}$). This contradicts to (3.5).

Step V. (i) Denote by $\mathcal{S}$ the plane $\mathrm{span}\{p_1, \ldots, p_{N-\widetilde{N}}\}$, where the vectors $p_1, \ldots, p_{N-\widetilde{N}}$ are defined in Step **II**, and consider $\mathcal{S} \cap \mathrm{Box}_2(0, r)$. Without loss of generality we may assume that $p_1, \ldots, p_{N-\widetilde{N}}$ are orthonormal (it suffices to carry out the procedure described in Step **II** within each $\mathcal{V}_i$, $i = 1, \ldots, M$). Verify that this intersection equals a direct product of balls of radii r^k, $k = 1, \ldots, M$, in some basis.

To this end, we consider a new orthonormal basis $\mathcal{O}e_1, \ldots, \mathcal{O}e_N$ in $\mathbb{R}^N$, where the orthogonal transformation $\mathcal{O}$ satisfies $\mathcal{O}(\mathrm{span}\{\mathcal{V}_i\}) = \mathrm{span}\{\mathcal{V}_i\}$, $i = 1, \ldots, M$, and the image of each p_j is the standard basis vector e_{i_j}, $j = 1, \ldots, N - \widetilde{N}$. Then the d_2-ball in the initial basis coincides with the one in the new metric: $\mathcal{O}(\mathrm{Box}_2(0, r)) = \mathrm{Box}_2(0, r)$. Thus, we have constructed a basis for $\mathcal{S}$ consisting of $N - \widetilde{N}$ standard basis vectors.

Next, $\mathcal{O}(\mathrm{Box}_2(0, r) \cap \mathcal{S}) = \mathrm{Box}_2(0, r) \cap \mathcal{O}(\mathcal{S}) = \mathrm{Box}_2^{\mathcal{O}(\mathcal{S})}(0, r)$, where the latter is a ball in the metric $d_2|_{\mathcal{O}(\mathcal{S})}$. By the choice of $\mathcal{O}$, it equals a direct product of Euclidean balls.

(ii) Obviously, the $(N - \widetilde{N})$-dimensional Lebesgue measure of $\mathrm{Box}_2^{\mathcal{O}(\mathcal{S})}(0, r)$ equals $C r^{\nu - \nu_0(x)}$, where C depends only on x. Since $\mathcal{O}$ is an orthogonal transformation, so is the inverse mapping $\mathcal{O}^{-1}$, and the $(N - \widetilde{N})$-dimensional Lebesgue measure of $\mathrm{Box}_2(0, r) \cap \mathcal{S} = \mathcal{O}^{-1}(\mathrm{Box}_2^{\mathcal{O}(\mathcal{S})}(0, r))$ is the same; thus, it also equals $C r^{\nu - \nu_0(x)}$.

Observe that at regular points the $(N - \widetilde{N})$-dimensional Lebesgue measure of $\mathcal{S} \cap \mathrm{Box}_2(0, r)$ equals $\prod_{k=1}^{M} \omega_{n_k - \tilde{n}_k} r^{\nu - \tilde{\nu}}$.

Step VI. On this step we show that there exists $r_0 > 0$ such that for $r \leq r_0$ the set

$$\pi_{\mathcal{T}}(\mathcal{T} \cap \mathrm{Box}_2(0, r))$$

coincides with the $o(r)$-neighborhood of $\mathcal{S} \cap \mathrm{Box}_2(0, r)$ in $\mathcal{S}$, where $o(r)$ is taken with respect to the metric d_2. From this, we deduce that the $\mathcal{H}^{N - \widetilde{N}}$-measure distortion of $\pi_{\mathcal{T}}$ equals

$$\lim_{r \to 0} \frac{\mathcal{H}^{N - \widetilde{N}}(\pi_{\mathcal{T}}(\mathcal{T} \cap \mathrm{Box}_2(0, r)))}{\mathcal{H}^{N - \widetilde{N}}(\mathcal{T} \cap \mathrm{Box}_2(0, r))} = \lim_{r \to 0} \frac{\mathcal{H}^{N - \widetilde{N}}(\mathcal{S} \cap \mathrm{Box}_2(0, r))}{\mathcal{H}^{N - \widetilde{N}}(\mathcal{T} \cap \mathrm{Box}_2(0, r))}.$$

(i) Fix a vector w_j. Show that there exists $r_{0j} > 0$ such that the length of $\mathbb{R}w_j \cap \mathrm{Box}_2(0, r)$ equals $O(r^{k(j)})$ for all $r \in (0, r_{0j})$. Indeed, fix $r_{0j} > 0$ and a take point $\bar{u} \in \mathbb{R}_+ w_j \cap \partial \mathrm{Box}_2(x, r_{0j})$ and the corresponding vector u whose coordinates coincide with those of $\bar{u}$. Here

$$\mathbb{R}_+ w_j = \{z \in \mathbb{R}^N : z = a w_j, a \in \mathbb{R}_+\}.$$

Then $u = \sum_{k=1}^{k(j)} u_k$, where

$$u_k = u_k(r_{0j}) \in \mathrm{span}\{\mathcal{V}_k\},$$

and $d_2(0, \bar{u}) = r_{0j}$. Consequently, $|u_k(r_{0j})| \leq (r_{0j})^k$; here $|\cdot|$ denotes the Euclidean length.

We verify that $|u_k(r)| = O(r^{k(j)})$ for all $r \in (0, r_{0j})$, $k = 1, \ldots, k(j)$. The vector $u(r) = \sum_{k=1}^{k(j)} u_k(r)$ has the same coordinates as the point $\mathbb{R}_+ w_j \cap \partial \operatorname{Box}_2(0, r)$. Assume on the contrary that for every r_{0j} and for every $K < \infty$ there exist $r < r_{0j}$ and $u_l(r)$ with

$$|u_l(r)| > K r^{k(j)}.$$

Fix such r_0, K and $r < r_0$, and put $\alpha_r = \frac{|u|}{|u(r)|}$. Then we can represent $u(r)$ as

$$u(r) = \alpha_r u = \sum_{k=1}^{k(j)} \alpha_r u_k = \sum_{k=1}^{k(j)} u_k(r).$$

On the one hand, $K r^{k(j)} < |u_l(r)| = |\alpha_r u_l| = |\alpha_r| |u_l|$. Therefore,

$$|\alpha_r| > K \frac{r^{k(j)}}{|u_l|}.$$

On the other hand, $k = k(j)$, in view of its definition, satisfies $|u_{k(j)}(r)| \le L r^{k(j)}$ for all $r \in (0, r_0]$. Since $u_{k(j)}(r) = \alpha_r \cdot u_{k(j)}$, we have

$$L r^{k(j)} \ge |u_{k(j)}(r)| = |\alpha_r| \cdot |u_{k(j)}| > K r^{k(j)} \frac{|u_{k(j)}|}{|u_l|}.$$

Observe that $\frac{|u_{k(j)}|}{|u_l|}$ depends only on w_j, and is independent of $r > 0$. Obviously, the equality

$$L r^{k(j)} > K r^{k(j)}$$

is violated for $K > L$. Thus we arrive at a contradiction. Consequently, $|u_k| = O(r^{k(j)})$, $r \in (0, r_0)$, for all $k = 1, \ldots, k(j)$.

(ii) Consider $\mathcal{T} \cap \operatorname{Box}_2(0, r)$, and the projection $\pi_{\mathcal{T}} : \mathcal{T} \to \mathcal{S}$. By the properties of the matrix W (see (3.3)), it is bijective. Verify that

$$d_2(0, \partial[\pi_{\mathcal{T}}(\mathcal{T} \cap \operatorname{Box}_2(0, r))]) = r(1 + o(1)), \tag{3.7}$$

where ∂ stands for the boundary relative to the plane $\mathcal{T}$. First, consider the basis elements w in $\mathcal{T}$. Represent each of them as $w = y + z$, where $\pi_{\mathcal{T}}(y + z) = y$. The properties of $\pi_{\mathcal{T}}$ yield $\deg z < \deg y$, where the degree of the vector $u = \sum_{i=1}^{\tilde{N}} u_i e_i$ is defined as

$$\deg u = \deg\left(\sum_{i=1}^{\tilde{N}} u_i X_i\right) = \max_{i=1,\ldots,N} \{\deg X_i : u_i \ne 0\}.$$

If $w \in \mathcal{T}$ is not a basis vector, then we expand it in the basis $\{w_i\}_{i=1}^{N-\widetilde{N}}$ as

$$w = \sum_{k=1}^{N-\widetilde{N}} a_k w_k = \sum_{k=1}^{N-\widetilde{N}} a_k y_k + \sum_{k=1}^{N-\widetilde{N}} a_k z_k,$$

where $\pi_{\mathcal{T}}(w_k) = y_k$, $k = 1, \ldots, N - \widetilde{N}$. Next, it suffices to note that

$$\deg\Big(\sum_{k=1}^{N-\widetilde{N}} a_k y_k\Big) > \deg\Big(\sum_{k=1}^{N-\widetilde{N}} a_k z_k\Big).$$

Thus, we can represent every $w \in \mathcal{T}$ as $w = y + z$, where $\pi_{\mathcal{T}}(w) = y$ and $\deg y > \deg z$.

Define a new quasimetric d_{2E}^0 on $\mathrm{Box}_2(0, r)$. For $v, w \in \mathrm{Box}_2(0, r)$ put $d_{2E}^0(v, w) = d_2(0, w - v)$, where $w - v$ denotes the Euclidean difference. This definition implies that $\mathrm{Box}_2(0, r)$ coincides with the ball $\mathrm{Box}_{2E}(0, r)$ of radius r centered at 0 in the metric d_{2E}^0. Consequently, for proving (3.7), it suffices to show that

$$d_{2E}^0(0, \partial[\pi_{\mathcal{T}}(\mathcal{T} \cap \mathrm{Box}_2(0, r))]) = r(1 + o(1)),$$

where ∂ stands for the boundary relative to the plane $\pi_{\mathcal{T}}(\mathcal{T})$. Take $w = y + z$ with $d_2(0, w) = r$ and $\deg y > \deg z$. Then

$$d_2(0, y + z) = d_{2E}^0(0, y + z) = r,$$

$\rho(0, y + z) = O(r^{\deg y})$ (see Step **V**, substep **(i)**), and $\rho(0, y) = O(r^{\deg y})$ since $\pi_{\mathcal{T}}$ is a linear mapping. Then, we have $\rho(y, y + z) = O(r^{\deg y})$ because $y \perp z$.

Observe that $d_{2E}^0(y, y + z) = d_2(0, z)$. Represent z as $z = \sum_{i=1}^{I(z)} z_i$, where $z_i \in \mathcal{V}_i$. Here $\deg z_{I(z)} = I(z) < \deg y$ because $\deg z < \deg y$. Since

$$O(r^{\deg y}) = \rho(y, y + z) \sim \max_{\substack{i=1,\ldots,I(z) \\ j=1,\ldots,\dim H_i/H_{i-1}}} \{|(z_i)_j|\},$$

it follows that $|z_i| \le O(r^{\deg y})$.

This implies that

$$d_2(0, z) = \max_{i=1,\ldots,I(z)} \{|z_i|^{\frac{1}{i}}\} \le \max_{i=1,\ldots,I(z)} \{|z_i|^{\frac{1}{I(z)}}\}$$

$$\le O(r^{\frac{\deg y}{I(z)}}) \le O(r^{1 + \frac{1}{M-1}}) = r \cdot o(1),$$

where $o(1)$ is at most $\mathcal{C}r^{\frac{1}{M-1}}$ for some $0 < \mathcal{C} < \infty$ independent of the point of $\mathrm{Box}_2(0, r)$.

Proposition 3.3 implies that $d_{2E}^0(v, w) \leq d_{2E}^0(v, u) + cd_{2E}^0(u, w)$ for $u, v, w \in \mathrm{Box}_2(0, r)$ and some $c > 0$ (it suffices to put $\widehat{F}_{\mu,\beta}^i(x) \equiv 0$ in Proposition 3.3). Consequently,

$$d_2(0, y) = d_{2E}^0(0, y) \leq d_{2E}^0(0, z + y) + cd_{2E}^0(y, z + y) = r(1 + o(1)),$$

and (3.7) follows.

Thus, the measure of $\mathcal{T} \cap \mathrm{Box}_2(x, r)$ is equivalent to $Cr^{\nu - \nu_0(x)}$ as $r \to 0$. The theorem follows. $\qquad\square$

Definition 3.9. Let $\xi : \mathbb{M} \to \widetilde{\mathbb{M}}$ be a mapping of two Carnot manifolds. Fix $x \in \mathbb{M}$. The d_2-*distortion of ξ at y with respect to x equals*

$$\frac{d_2(\xi(y), \xi(x))}{d_2(y, x)},$$

and the ρ-*distortion equals*

$$\frac{\rho(\xi(y), \xi(x))}{\rho(y, x)}.$$

To simplify notation, we denote the Gram determinant $\sqrt{\det(AA^*)}$ for a matrix A by $\mathcal{D}(A)$. We denote the Gram determinant $\sqrt{\det(B^*B)}$ for a matrix B by $\widetilde{\mathcal{D}}(B)$.

Proposition 3.10. *The matrix of the differential of the mapping θ_x, $x \in \mathbb{M}$, at 0 equals the identity matrix.*

Henceforth we denote the Riemann tensor at y by $g(y)$.

Theorem 3.11. *Suppose that $x \in \varphi^{-1}(t)$ is a regular point. Then*

(I) *In a neighborhood of $0 = \theta_x^{-1}(x)$ there exists a mapping from $T_0[\psi^{-1}(t)] \cap \mathrm{Box}_2(0, r(1+o(1)))$ to $\psi^{-1}(t) \cap \mathrm{Box}_2(0, r)$ such that both d_2- and ρ-distortions with respect to 0 are equal to $1 + o(1)$ at every $y \in T_0[\psi^{-1}(t)] \cap \mathrm{Box}_2(0, r(1+o(1)))$, where $o(1)$ is uniform in $x = \theta_x(0) \in \mathcal{U} \Subset \mathbb{M}$ and in $y \in T_0[\psi^{-1}(t)] \cap \mathrm{Box}_2(0, r(1+o(1)))$.*

(II) *The $\mathcal{H}^{N-\widetilde{N}}$-measure of $\varphi^{-1}(t) \cap \mathrm{Box}_2(x, r)$ equals*

$$\widetilde{\mathcal{D}}(g|_{\ker D\varphi(x)}) \cdot \prod_{k=1}^{M} \omega_{n_k - \tilde{n}_k} \cdot \frac{\mathcal{D}(D\varphi(x))}{\mathcal{D}(\widehat{D}\varphi(x))} r^{\nu - \tilde{\nu}}(1 + o(1))$$

with $o(1) \to 0$ as $r \to 0$, where $o(1) \to 0$ uniformly in $x \in \mathcal{U} \Subset \mathbb{M}$.

In the proof of Theorem 3.11 we use the following notation introduced in Theorem 3.7. We denote the mapping $\varphi \circ \theta_x$ by ψ, and the tangent space $T_0[\psi^{-1}(t)]$ by $\mathcal{T}$. We also use the mapping $\pi_{\mathcal{T}}$ defined in Step **III** and the image $\mathcal{S} = \pi_{\mathcal{T}}(\mathcal{T})$.

Proof of Theorem 3.11. Notice that without loss of generality we may assume that $\widehat{D}\psi(z)$ is strictly separated from 0 on $(\ker \widehat{D}\psi(0))^{\perp}$ for $z \in \mathrm{Box}_2(0, r)$, i. e., $\widehat{D}\psi(z)(v) \geq \alpha > 0$ for all $z \in \mathrm{Box}_2(0, r)$ and $v \in (\ker \widehat{D}\psi(0))^{\perp}$ with $|v| = 1$ (here $|\cdot|$ denotes the Euclidean norm).

Step I. Verify that C in (3.2) at a regular point equals $\frac{\mathcal{D}(D\varphi(x))}{\mathcal{D}(\widehat{D}\varphi(x))}$ up to a Riemannian factor. In particular, we show that $\frac{\mathcal{D}(D\varphi(x))}{\mathcal{D}(\widehat{D}\varphi(x))}$ equals the measure distortion under the mapping $\pi_{\mathcal{T}}$ defined in Step **II** of Theorem 3.7.

To this end, consider the projection of the normal space $\mathcal{N}$ to $\mathcal{T}$ onto $H_{\widetilde{M}}(0)$ constructed by the choice of vectors written as rows in the matrix of $\widehat{D}\psi(0)$ (recall that the matrix of $D\psi(0)$ equals the matrix of $([(\theta_x^{-1})_* X_1]\psi, \ldots, [(\theta_x^{-1})_* X_N]\psi)$, and in the bases $\{e_i\}_{i=1}^N$ and $\{e_i\}_{i=1}^{\widetilde{N}}$ the matrices of $D\psi(0)$ and $\widehat{D}\psi(0)$ have the structure similar to (2.4) and (2.5), respectively). Denote this projection by $\pi_{\mathcal{N}}$ and its image by $\mathcal{L} = \pi_{\mathcal{N}}(\mathcal{N})$. Verify that $\mathcal{S}$ is orthogonal to $\mathcal{L}$. Write a basis for $\mathcal{N}$ as the row vectors of the matrix $D\psi(0)$ in the basis $\{e_i\}_{i=1}^N$.

(i) Without loss of generality we may assume that $\pi_{\mathcal{N}}$ is an orthogonal projection (see Theorem 3.7). Indeed, the matrix of $D\psi(0)$ admits "gradings" $\widetilde{\mathcal{B}}_1, \ldots, \widetilde{\mathcal{B}}_{\widetilde{M}}$ of the columns and $\widetilde{\mathcal{A}}_1, \ldots, \widetilde{\mathcal{A}}_{\widetilde{M}}$ of the rows similar to those of the matrix Δ of Step **II** substep **(ii)** of Theorem 3.7. Namely, we say that a column vector belongs to the block $\widetilde{\mathcal{B}}_k$ if its index is at least $\dim H_{k-1} + 1$ (here we assume that $\dim H_0 = 0$) and at most $\dim H_k$, $k = 1, \ldots, \widetilde{M}$, and we say that a row vector belongs to the block $\widetilde{\mathcal{A}}_l$ if the index of its first nonzero element is at least $\dim H_{k-1} + 1$ (here we assume that $\dim H_0 = 0$) and at most $\dim H_k$, $k = 1, \ldots, \widetilde{M}$. Put $\widetilde{\mathcal{V}}_k = \widetilde{\mathcal{B}}_k \cap \widetilde{\mathcal{A}}_k$, $k = 1, \ldots, \widetilde{M}$.

Next, we transform the blocks $\widetilde{\mathcal{B}}_k \setminus \widetilde{\mathcal{V}}_k$, $k = 1, \ldots, \widetilde{M}$. Define the required transformation of the matrix of $D\psi(0)$ by induction. For $k = 1$, we have nothing to transform; thus, the base of induction holds. Suppose that we have transformed the blocks $\widetilde{\mathcal{B}}_k \setminus \widetilde{\mathcal{V}}_k$, $k = 1, \ldots, l$, $l \geq 1$. Assume that $\widetilde{\mathcal{V}}_{l+1} \neq \varnothing$ (otherwise, the transformation is trivial). Replace the blocks $\widetilde{\mathcal{A}}_1, \ldots, \widetilde{\mathcal{A}}_l$ by the projections of their row vectors onto the space

$$(\mathrm{span}\{\mathcal{A}_{l+1}\})^{\perp} \cap \mathrm{span}\{\widetilde{\mathcal{A}}_1, \ldots, \widetilde{\mathcal{A}}_{l+1}\}.$$

This projection leaves the blocks $\widetilde{\mathcal{B}}_k \setminus \widetilde{\mathcal{V}}_k$, $k = 1, \ldots, l$, $l \geq 1$, unchanged because of the "triangular" structure of the matrix of $D\psi(0)$.

Moreover, the rows of the block $\widetilde{\mathcal{B}}_k \setminus \mathcal{V}_k$ are orthogonal to those of $\widetilde{\mathcal{V}}_k$, $k = 1, \ldots, \widetilde{M}$. It results

$$
\begin{pmatrix}
\mathcal{V}_1 & \mathcal{V}_2^\perp & \mathcal{V}_3^\perp & * & * & & \vdots & \vdots & * & \ldots & * \\
0 & \mathcal{V}_2 & & \vdots & \vdots & * & & * & * & \ldots & * \\
0 & 0 & \mathcal{V}_3 & & & * & \mathcal{V}_{\widetilde{M}-1}^\perp & \mathcal{V}_M^\perp & * & \ldots & * \\
\vdots & \vdots & 0 & 0 & \ddots & \ddots & \ddots & * & & * & \ldots & * \\
0 & 0 & & \vdots & 0 & 0 & 0 & \mathcal{V}_{\widetilde{M}-1} & & * & \ldots & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{V}_{\widetilde{M}} & * & \ldots & *
\end{pmatrix}
$$

By the construction, the blocks $\mathcal{V}_1, \ldots, \mathcal{V}_{\widetilde{M}}$ constitute the matrix of $\widehat{D}\psi(0)$ (see (2.5)), and $\dim \mathcal{L} = \widetilde{N}$ at regular points.

(ii) It is easy to see that $\mathcal{L}$ is orthogonal to $\mathcal{S}$.

Indeed, take two vectors $v \in \mathcal{L}$ and $w \in \mathcal{S}$. First, suppose that v and w are the images of arbitrary basis vectors in $\mathcal{N}$ and $\mathcal{T}$ respectively. Then $v \in \widetilde{\mathcal{V}}_i$ and $w \in \mathcal{V}_j$ for some i and j. If $i \neq j$ then obviously $v \perp w$ (since they have no corresponding nonzero components).

Suppose that $i = j$ and consider the preimages of v and w. By construction, they are orthogonal. Note that the preimage of v belongs to $\mathcal{N}$, and we can write it as $(0, \ldots, 0, \tilde{v}, *)$, where the nonzero part begins with some element $\tilde{v}$ of v. The preimage of w belongs to $\mathcal{T}$, and it can be written as $(*, \tilde{w}, 0, \ldots, 0)$, where the last component of the nonzero part is some element $\tilde{w}$ of w. Consequently,

$$
0 = (0, \ldots, 0, \tilde{v}, *) \cdot (*, \tilde{w}, 0, \ldots, 0)^T = (v, w),
$$

and $v \perp w$.

Since v and w are the images of arbitrary basis vectors in $\mathcal{T}$ and $\mathcal{N}$, we have $\mathcal{S} \perp \mathcal{L}$.

(iii) Thus, $\mathcal{S}$ is orthogonal to $\mathcal{L}$. Moreover, the projection of $\mathcal{T}$ onto $\mathcal{S}$ is also orthogonal (see Theorem 3.7): $(v - \pi_{\mathcal{T}}(v)) \perp \pi_{\mathcal{T}}(v)$ for all $v \in \mathcal{T}$. Let us verify that the measure distortions under the projections $\pi_{\mathcal{T}}$ and $\pi_{\mathcal{N}}$ coincide.

Step II. (i) Show that

$$
\dim \mathcal{T} - \dim(\mathcal{T} \cap \mathcal{S}) = \dim \mathcal{N} - \dim(\mathcal{N} \cap \mathcal{L}).
$$

Put $l = \dim(\mathcal{T} \cap \mathcal{S})$. On the one hand, since $\mathcal{N} \perp \mathcal{T}$ and $\mathcal{L} \perp \mathcal{S}$, it follows that $\operatorname{span}\{\mathcal{N}, \mathcal{L}\} \perp (\mathcal{T} \cap \mathcal{S})$, and so $\operatorname{span}\{\mathcal{N}, \mathcal{L}\} \subset (\mathcal{T} \cap \mathcal{S})^\perp$. On the other hand, if $v \perp (\mathcal{T} \cap \mathcal{S})$ then $v \in \operatorname{span}\{\mathcal{N}, \mathcal{L}\}$. Indeed, assume on the contrary

that $v \notin \mathrm{span}\{\mathcal{N}, \mathcal{L}\}$. Then $v = v_{\mathcal{N}\mathcal{L}} + v_{(\mathcal{N}\mathcal{L})^\perp}$, where $v_{\mathcal{N}\mathcal{L}} \in \mathrm{span}\{\mathcal{N}, \mathcal{L}\}$ and $v_{(\mathcal{N}\mathcal{L})^\perp} \in (\mathrm{span}\{\mathcal{N}, \mathcal{L}\})^\perp$ with $v_{(\mathcal{N}\mathcal{L})^\perp} \neq 0$. Consequently, since $v_{(\mathcal{N}\mathcal{L})^\perp} \perp \mathcal{N}$ and $v_{(\mathcal{N}\mathcal{L})^\perp} \perp \mathcal{L}$, we obtain $v_{(\mathcal{N}\mathcal{L})^\perp} \in (\mathcal{T} \cap \mathcal{S})$ and arrive at a contradiction.

Thus, $\mathrm{span}\{\mathcal{N}, \mathcal{L}\} = (\mathcal{T} \cap \mathcal{S})^\perp$, and $\dim \mathrm{span}\{\mathcal{N}, \mathcal{L}\} = N - l$. Since $\dim \mathcal{N} = \dim \mathcal{L}$, we get

$$\dim(\mathcal{N} \cap \mathcal{L}) = 2 \dim \mathcal{N} - \dim \mathrm{span}\{\mathcal{N}, \mathcal{L}\} = 2\widetilde{N} - N + l.$$

Consequently,

$$\dim \mathcal{T} - \dim(\mathcal{T} \cap \mathcal{S}) = N - \widetilde{N} - l = \dim \mathcal{N} - \dim(\mathcal{N} \cap \mathcal{L}). \qquad (3.8)$$

(ii) Recall that $\mathcal{L} = \ker \widehat{D}\psi(0)$ at regular points (see Step **I**, substep **(ii)**). Verify that $\dim(\mathcal{T} \cap \mathcal{L}) = \dim(\mathcal{N} \cap \mathcal{S}) = 0$. Indeed, assume on the contrary that there exists $h \in \mathcal{T} \cap \mathcal{L}$, $h \neq 0$. Then, $h \perp \mathcal{N}$ and $h \perp \mathcal{S}$. Since $(\mathcal{T} \cap \mathcal{L})^\perp = \mathrm{span}\{\mathcal{N}, \mathcal{S}\}$, which we can justify similarly to the substep **(i)** above, it follows that $\dim \mathrm{span}\{\mathcal{N}, \mathcal{S}\} \leq N - 1$. Consequently, from $\dim \mathrm{span}\{\mathcal{N}\} = \widetilde{N}$ and $\dim \mathrm{span}\{\mathcal{S}\} = N - \widetilde{N}$, we infer that $\dim(\mathcal{N} \cap \mathcal{S}) \geq 1$ and there exists $h^\perp \in (\mathcal{N} \cap \mathcal{S})$; thus, $\widehat{D}\psi(0)$ vanishes on some vector in $\mathcal{N}$. This implies that one of $\widetilde{N}$ vectors in $\mathcal{N}$ is orthogonal to $\mathcal{L}$. Consequently, its image under $\pi_\mathcal{N}$ vanishes since every vector in $\mathcal{N}$ can be uniquely represented as a sum of a vector in $\mathcal{L}$ and a vector in $\mathcal{L}^\perp$. Thus, $\pi_\mathcal{N}$ has a nonzero kernel, and $\dim(\pi_\mathcal{N}(\mathcal{N})) < \widetilde{N}$. This implies that $\dim[\pi_\mathcal{N}(\mathcal{N})] = \dim[(\ker \widehat{D}\psi(0))^\perp] < \widetilde{N}$, which contradicts the regularity of x.

Hence, $\dim(\mathcal{T} \cap \mathcal{L}) = 0$ and $\dim(\mathcal{N} \cap \mathcal{S}) = 0$.

(iii) Put $q = \dim \mathcal{T} - \dim(\mathcal{T} \cap \mathcal{S})$. Consider the maximal subset $\mathcal{S}'$ of $\mathcal{S}$ satisfying $\mathcal{S}' \perp (\mathcal{T} \cap \mathcal{S})$. Namely, $\mathcal{S}' = (\mathcal{T} \cap \mathcal{S})^\perp \cap \mathcal{S}$. Then $\dim \mathcal{S}' = q$. Put $\mathcal{T}' = \pi_\mathcal{T}^{-1}(\mathcal{S}')$. It is easy to see that $\mathcal{T}' \perp (\mathcal{T} \cap \mathcal{S})$. Indeed, take arbitrary vectors $v \in \mathcal{T} \cap \mathcal{S}$, $v' \in \mathcal{S}'$, and a vector w, such that $v' + w = \pi_\mathcal{T}^{-1}(v')$. In particular, $v' + w \in \mathcal{T}$. Note that $w \perp (\mathcal{T} \cap \mathcal{S})$ because $\pi_\mathcal{T}$ is an orthogonal projection. Then $\mathcal{T} \ni \pi_\mathcal{T}^{-1}(v') = (v' + w) \perp v$. Since v and v' are arbitrary, it follows that $\mathcal{T}' \perp (\mathcal{T} \cap \mathcal{S})$. The non-degeneracy of $\pi_\mathcal{T}$ and $\pi_\mathcal{T}^{-1}$ yields $\dim \mathcal{T}' = q$.

Thus, we may regard the following mapping as the "inverse" projection $\pi_\mathcal{T}^{-1}$. To each $v \in \mathcal{S}$ it assigns the vector $w \in \mathrm{span}\{v, \mathcal{L}\} \cap \mathcal{T}$ such that $w = (v + \mathcal{L}) \cap \mathcal{T}$. It is well defined since $\mathcal{L} \perp \mathcal{S}$ and $\dim(\mathcal{T} \cap \mathcal{L}) = 0$.

(iv) Since $\mathcal{L} \cap \mathcal{N}$ is orthogonal to both $\mathcal{S}$ and $\mathcal{T}$, we have

$$(v + \mathcal{L}) \cap \mathcal{T} = (v + \mathcal{E}) \cap \mathcal{T}$$

for $v \in \mathcal{S}$, where $\mathcal{E} \subset (\mathcal{L} \cap \mathcal{N})^\perp \cap \mathcal{L}$ is the minimal subset enjoying this property. Let us prove that $\mathcal{E} = (\mathcal{L} \cap \mathcal{N})^\perp \cap \mathcal{L}$. It is easy to see that if

$w = (v+\mathcal{L}) \cap \mathcal{T} = v+u$ with $u \in \mathcal{L}$ then $u = w-v \in \mathrm{span}\{\mathcal{S},\mathcal{T}\} = (\mathcal{L} \cap \mathcal{N})^{\perp}$. Consequently, $u \in (\mathcal{L} \cap \mathcal{N})^{\perp} \cap \mathcal{L}$, and $\dim \mathcal{E} \leq N - \widetilde{N} - l$.

Verify that $\dim \mathcal{E} \geq N - \widetilde{N} - l$. Assume on the contrary that $\mathcal{E} \neq (\mathcal{L} \cap \mathcal{N})^{\perp} \cap \mathcal{L}$, and consider $v_0 \in \mathcal{E}^{\perp} \cap ((\mathcal{L} \cap \mathcal{N})^{\perp} \cap \mathcal{L})$. Since $v_0 \in \mathrm{span}\{\mathcal{T},\mathcal{S}\} = (\mathcal{L} \cap \mathcal{N})^{\perp}$, it follows that $v_0 = v_0^{\mathcal{T}} + v_0^{\mathcal{S}}$, where $v_0^{\mathcal{T}} \in \mathcal{T}$ and $v_0^{\mathcal{S}} \in \mathcal{S}$. Take $v = -v_0^{\mathcal{S}} \in \mathcal{S}$ and consider $v_0^{\mathcal{T}} = v + v_0 \in \mathcal{T}$. By the assumption on $\mathcal{E}$ and because $\pi_{\mathcal{T}}$ is an epimorphism, $v + v_0 = v^{\mathcal{S}} + v^{\mathcal{E}}$, where $v^{\mathcal{S}} \in \mathcal{S}$ and $v^{\mathcal{E}} \in \mathcal{E}$. Hence, $v - v^{\mathcal{S}} = v^{\mathcal{E}} - v_0$, where $v - v^{\mathcal{S}} \in \mathcal{S}$ and $v^{\mathcal{E}} - v_0 \in \mathcal{L}$. Since $\mathcal{L} \perp \mathcal{S}$, we arrive at a contradiction.

(v) Similarly, we show that $\pi_{\mathcal{N}}^{-1}(w) = (w + \mathcal{F}) \cap \mathcal{N}$, where $\mathcal{F} = (\mathcal{S} \cap \mathcal{T})^{\perp} \cap \mathcal{S} = \mathcal{S}'$, and $\dim \mathcal{F} = \dim \mathcal{E}$.

(vi) Take a cube of radius r in $\mathcal{S}$ which is the direct product of a cube of radius r in $\mathcal{T} \cap \mathcal{S}$ and a cube of radius r in $\mathcal{S}'$. Then its image equals the direct product of a cube of radius r in $\mathcal{T} \cap \mathcal{S}$ and a subset of $\mathcal{T}$. Hence, it suffices to calculate the measure distortion of $\pi_{\mathcal{T}}^{-1}$ on $\mathcal{S}'$. Similarly, we infer that it suffices to calculate the measure distortion of $\pi_{\mathcal{N}}^{-1}$ on $\mathcal{L}'$.

(vii) Consider the orthogonal mapping Θ that is a rotation about $\mathrm{span}\{\mathcal{N} \cap \mathcal{L}, \mathcal{T} \cap \mathcal{S}\}$ such that

- $\Theta(\mathcal{F}) = \mathcal{E}$

- $\Theta((\mathcal{N} \cap \mathcal{L})^{\perp} \cap \mathcal{L}) = (\mathcal{T} \cap \mathcal{S})^{\perp} \cap \mathcal{S}$.

Note that if we choose an orthonormal basis in $\mathbb{R}^N$ in the following order: $\mathrm{span}\{\mathcal{N} \cap \mathcal{L}, \mathcal{T} \cap \mathcal{S}\}$, $\mathrm{span}\{\mathcal{F}\}$, $\mathrm{span}\{(\mathcal{N} \cap \mathcal{L})^{\perp} \cap \mathcal{L}\}$ then the matrix of Θ looks like

$$\Theta = \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & E \\ 0 & \widetilde{E} & 0 \end{pmatrix},$$

where

$$\widetilde{E} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Consequently, $\pi_{\mathcal{T}}^{-1}|_{\mathcal{S}'}(B(0,r) \cap \mathcal{S}') = \Theta \circ \pi_{\mathcal{N}}^{-1}|_{\mathcal{L}'}(B(0,r) \cap \mathcal{L}')$ for any $r > 0$, and the measure distortions of these mappings coincide.

(viii) Thus, the measure distortions of $\pi_{\mathcal{N}}$ and $\pi_{\mathcal{T}}$ coincide.

Since both the Riemannian differential and hc-differential of θ_x at x equal identity, by the chain rule the corresponding "determinants" coincide:

$$\mathcal{D}(D\varphi(x)) = \mathcal{D}(D\psi(0)) \quad \text{and} \quad \mathcal{D}(\widehat{D}\varphi(x)) = \mathcal{D}(\widehat{D}\psi(0)).$$

Thus, the measure distortion under $\pi_{\mathcal{T}}^{-1}$ equals

$$\frac{\mathcal{D}(D\psi(x))}{\mathcal{D}(\widehat{D}\psi(x))} = \frac{\mathcal{D}(D\varphi(x))}{\mathcal{D}(\widehat{D}\varphi(x))}.$$

Step III. Verify that $\psi^{-1}(t) \cap \mathrm{Box}_2(0, r)$ is a subset of an $o(r)$-neighborhood of $\ker \widehat{D}\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1)))$ with respect to d_{2E}^0. To this end, we prove that $\partial(\psi^{-1}(t) \cap \mathrm{Box}_2(0, r))$ is a subset of the $o(r)$-neighborhood of $\partial(\ker \widehat{D}\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1))))$ with respect to d_2, where $o(1)$ is uniform in $x = \theta_x(0)$, $x \in \mathcal{U} \Subset \mathbb{M}$.

Indeed, consider $y \in \partial(\psi^{-1}(t) \cap \mathrm{Box}_2(0, r))$. We can represent this point as $y = y_{\widehat{D}\psi} + y_{\ker \widehat{D}\psi}$, where $y_{\widehat{D}\psi} \in (\ker \widehat{D}\psi(0))^\perp$ and $y_{\ker \widehat{D}\psi} \in \ker \widehat{D}\psi(0)$. Then the definition of the hc-differentiability implies that

$$o(r) = d_2^{\psi(0)}(\widehat{D}\psi(0)(y), \psi(y)) = d_2^{\psi(0)}(\widehat{D}\psi(0)(y_{\widehat{D}\psi}), \psi(y))$$
$$= d_2^{\psi(0)}(\widehat{D}\psi(0)(y_{\widehat{D}\psi}), \psi(0)).$$

We have $d_2^{\psi(0)}(\psi(0), \widehat{D}\psi(0)(y_{\widehat{D}\psi})) = o(r)$, and $d_2^0(0, y_{\widehat{D}\psi}) = o(r)$, where $o(\cdot)$ is uniform in y, $x = \theta_x(0)$, $x \in \mathcal{U} \Subset \mathbb{M}$. The inequality

$$d_{2E}^0(0, u + v) \leq d_{2E}^0(0, u) + c d_{2E}^0(0, v) \qquad (3.9)$$

implies that

$$d_{2E}^0(0, y_{\ker \widehat{D}\psi}) \leq d_{2E}^0(0, y) + c d_{2E}^0(0, y_{\widehat{D}\psi}) = r + o(r),$$

and hence, $d_{2E}^0(0, y_{\ker \widehat{D}\psi}) = r(1 + o(1))$ and $d_2^0(0, y_{\ker \widehat{D}\psi}) = r(1 + o(1))$. Thus, $\partial(\psi^{-1}(t) \cap \mathrm{Box}_2(0, r))$ is a subset of the $o(r)$-neighborhood of $\partial(\ker \widehat{D}\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1))))$ with respect to d_2.

Step IV. Similarly considering the linear mapping $L(y) = D\psi(0)y$, we infer that $\partial(\ker D\psi(0) \cap \mathrm{Box}_2(0, r))$ is a subset of the $o(r)$-neighborhood of $\partial(\ker \widehat{D}\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1))))$ with respect to d_{2E}^0. Indeed, we have $\ker D\psi(0) = L^{-1}(0)$ and $DL(0) = D\psi(0)$, and L is hc-differentiable at 0 since $D\psi(0)$ is horizontal, and $\widehat{D}L(0) = \widehat{D}\psi(0)$.

Step V. Since there exists a bijective linear mapping from $\ker \widehat{D}\psi(0) \cap \mathrm{Box}_2(0, r)$ to $\ker D\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1)))$, by Step **IV** we infer that $\partial(\ker \widehat{D}\psi(0) \cap \mathrm{Box}_2(0, r))$ is a subset of the $o(r)$-neighborhood of $\partial(\ker D\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1))))$.

Hence, $\partial(\psi^{-1}(t) \cap \mathrm{Box}_2(0, r))$ is a subset of the $o(r)$-neighborhood of $\partial(\ker D\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1))))$ with respect to d_{2E}^0.

Step VI. In Steps **VII** – **IX** we explain that

$$\partial(\ker D\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1))))$$

is a subset of the $o(r)$-neighborhood of $\partial(\psi^{-1}(t) \cap \mathrm{Box}_2(0, r))$ with respect to d_{2E}^0. Moreover, similar arguments imply that the same is true regarding the sets $\ker D\psi(0) \cap \mathrm{Box}_2(0, r(1 + o(1)))$ and $\psi^{-1}(t) \cap \mathrm{Box}_2(0, r)$. Indeed, it suffices to recall that $o(\cdot)$ in the previous steps are uniform in $r > 0$ and $x = \theta_x(0)$.

Step VII. In order to justify the result of Step **VI**, we construct a one-to-one mapping from $\partial(\psi^{-1}(t) \cap \mathrm{Box}_2(x, r))$ to a subset of the $o(r)$-neighborhood of $\partial(\ker D\psi(0) \cap \mathrm{Box}_2(x, r(1 + o(1))))$ (with respect to d_{2E}^0) lying in the plane $\ker D\psi(0)$.

First of all, observe that at the regular point 0 we have $\ker D\psi \cap (\ker \widehat{D}\psi)^\perp = \{0\}$ by Step **II**, substep **(iii)**.

From this we deduce the following assumption: without loss of generality we may assume that $D\psi(z)(v) \geq \beta$, where $z \in U \supset \mathrm{Box}_2(0, r)$, $\beta > 0$, for $v \in (\ker \widehat{D}\psi(0))^\perp$ with $|v| = 1$.

This implies that in a neighborhood of a regular point we can expand each point y uniquely as $y = y_{\ker D\psi} + y_{\widehat{D}\psi}$, where $y_{\ker D\psi} \in \ker D\psi$ and $y_{\widehat{D}\psi} \in (\ker \widehat{D}\psi)^\perp$. Suppose that $y \in \psi^{-1}(t) \cap \partial\mathrm{Box}_2(0, r)$ and consider the mapping $\xi : y \mapsto y_{\ker D\psi}$.

Verify that $d_2(0, \xi(y)) = r(1 + o(1))$. To this end, we prove that $d_2^0(0, y - \xi(y)) = o(r)$ and then apply (3.9) for d_{2E}^0.

Represent y as $y = y_{\ker D\psi} + y_{\widehat{D}\psi} = v_{\ker \widehat{D}\psi} + v_{\widehat{D}\psi} + y_{\widehat{D}\psi}$, where $y_{\ker D\psi} = v_{\ker \widehat{D}\psi} + v_{\widehat{D}\psi}$. Since $d_{2E}^0(0, y) = r$, Step **III** implies that $d_{2E}^0(0, v_{\ker \widehat{D}\psi}) = r + o(r)$ and $d_{2E}^0(0, v_{\widehat{D}\psi} + y_{\widehat{D}\psi}) = o(r)$. Moreover, by Step **IV** we have

$$d_{2E}^0(0, y_{\ker D\psi}) = (1 + o(1))d_{2E}^0(0, v_{\ker \widehat{D}\psi}), \quad d_{2E}^0(0, v_{D\psi}) = o(1)d_{2E}^0(0, v_{\ker \widehat{D}\psi}),$$

and taking into account the generalized triangle inequality for d_{2E}^0, we conclude that $d_{2E}^0(0, y_{\widehat{D}\psi}) = o(r)$. Finally, we have $d_2^0(0, y - \xi(y)) = o(r)$.

Step VIII. On this step, we show that ξ is a bijective mapping. It is easy to see that ξ is injective. Indeed, it follows from $D\psi\langle(\ker \widehat{D}\psi)^\perp \cap \mathbb{S}^{N-1}\rangle \geq \beta > 0$.

To verify that it is also surjective, we show that the mapping $\eta : \psi^{-1} \cap \mathrm{Box}_2(0, 2r) \to \ker D\psi(0)$ is bi-Lipschitz. Here, η assigns to each $y \in \psi^{-1}(t) \cap \mathrm{Box}_2(0, r)$, the vector $y_{\ker D\psi}$, where $y = y_{\ker D\psi} + y_{\widehat{D}\psi}$, $y_{\ker D\psi} \in \ker D\psi$, and $y_{\widehat{D}\psi} \in (\ker \widehat{D}\psi)^\perp$. It is clear that $\eta|_{\psi^{-1}(t) \cap \partial\mathrm{Box}_2(0, r)} = \xi$.

Since η is a projection, it is Lipschitz with the Lipschitz constant being equal to 1. Verify that $|\eta(y_1) - \eta(y_2)| \geq K|y_1 - y_2|$ for some $K > 0$ for all $y_1, y_2 \in \psi^{-1}(t) \cap \mathrm{Box}_2(0, 2r)$.

Suppose that it is not so; thus, for every $\varepsilon > 0$ there exist $y_1, y_2 \in \psi^{-1}(t) \cap$ $\text{Box}_2(0, 2r)$ such that $|\eta(y_1) - \eta(y_2)| < \varepsilon |y_1 - y_2|$. Since $y_i = y_{i \ker D\psi} + y_{i \widehat{D}\psi}$, $y_{i \ker D\psi} \in \ker D\psi$ and $y_{i \widehat{D}\psi} \in (\ker \widehat{D}\psi)^\perp$, $i = 1, 2$, we infer that

$$|\eta(y_1) - \eta(y_2)| < \varepsilon |y_1 - y_2| \le \varepsilon |\eta(y_1) - \eta(y_2)| + \varepsilon |y_{1 \widehat{D}\psi} - y_{2 \widehat{D}\psi}|,$$

and

$$|y_{1 \widehat{D}\psi} - y_{2 \widehat{D}\psi}| \ge \frac{1 - \varepsilon}{\varepsilon} |\eta(y_1) - \eta(y_2)|.$$

Since $\ker D\psi(0) \cap (\ker \widehat{D}\psi(0))^\perp = 0$, the norm $|y_1 - y_2|_\eta = |\eta(y_1) - \eta(y_2)| + |y_{1 \widehat{D}\psi} - y_{2 \widehat{D}\psi}|$ is equivalent to the Euclidean norm. Furthermore,

$$\psi(y_1) = \psi(y_2) + D\psi(y_2)(y_1 - y_2) + o(|y_1 - y_2|),$$

and $D\psi(y_2)(y_1 - y_2) = o(|y_1 - y_2|)$. Since η is a linear mapping, we have $y_{1 \widehat{D}\psi} - y_{2 \widehat{D}\psi} = (y_1 - y_2)_{\widehat{D}\psi}$ and $\eta(y_1) - \eta(y_2) = \eta(y_1 - y_2)$.

On the one hand, by the definition of η we have

$$D\psi(y_2)(y_1 - y_2) = D\psi(0)(y_1 - y_2) + o(|y_1 - y_2|)$$
$$= D\psi(0)((y_1 - y_2)_{\widehat{D}\psi}) + o(|y_1 - y_2|),$$

and $D\psi(0)((y_1 - y_2)_{\widehat{D}\psi}) = o(|y_1 - y_2|)$. On the other hand,

$$|y_{1 \widehat{D}\psi} - y_{2 \widehat{D}\psi}| \ge \left(1 - \left(\frac{1 - \varepsilon}{\varepsilon} + 1\right)^{-1}\right) |y_1 - y_2|_\eta \ge L \left(1 - \left(\frac{1 - \varepsilon}{\varepsilon} + 1\right)^{-1}\right) |y_1 - y_2|,$$

where $L > 0$ depends only on U.

These relations lead to a contradiction with

$$D\psi(0) \langle (\ker \widehat{D}\psi(0))^\perp \rangle \ge \beta > 0.$$

Thus, η is a bi-Lipschitz mapping, and each $y \in \psi^{-1}(t) \cap \partial \text{Box}_2(0, r)$ has a unique preimage. Thus, ξ is bijective.

Step IX. Step **VIII** implies that ξ is invertible, and the previous steps imply that the d_2-distortion of ξ^{-1} is $(1 + o(1))$ with respect to 0. The estimate for $d_2(0, y - \eta^{-1}(y))$ is proved in Step **VII**. Observe that $o(1)$ depends on the convergence of $o(1)$ to 0 in the equality

$$d_2^{\psi(0)}(\widehat{D}\psi(0)y, \psi(y)) = o(1) \cdot d_2(0, y).$$

Thus, $o(1)$ is uniform in y. Consequently, $\psi^{-1}(t) \cap \partial \text{Box}_2(0, r)$ is a subset of the $o(r)$-neighborhood of $\ker D\psi(0) \cap \partial \text{Box}_2(0, r(1 + o(1)))$ with respect to d_2, and conversely, $\ker D\psi(0) \cap \partial \text{Box}_2(0, r)$ is a subset of the $o(r)$-neighborhood

37

of $\psi^{-1}(t) \cap \partial \operatorname{Box}_2(0,r)$ with respect to d_{2E}^0 (see Step **VII**). The same is true regarding the sets $\psi^{-1}(t) \cap \operatorname{Box}_2(0,r)$ and $\ker D\psi(0) \cap \operatorname{Box}_2(0, r(1+o(1)))$.

Note that the extension η of ξ onto $\psi^{-1}(t) \cap \operatorname{Box}_2(0,r)$ has the d_2-distortion with respect to 0 equal to $1 + o(1)$ as well. In Step **VIII**, we prove that this extension is bijective.

Step X. On this step, we show that the classical, that is, "Riemannian" distortion of η (and of ξ) is also $1 + o(1)$.

Indeed, since $\ker D\psi(0) \cap (\ker \widehat{D}\psi(0))^\perp = \{0\}$ and $D\psi\langle (\ker \widehat{D}\psi(0))^\perp \rangle \geq \beta > 0$ on U, it is easy to see that $\rho(0, y_{\widehat{D}\psi}) = o(\rho(0,y))$. Hence, $\rho(0, y_{\widehat{D}\psi}) = o(\rho(0, y_{\ker D\psi}))$, and the ρ-distortion of both η and η^{-1} equals $1 + o(1)$.

Since $o(\cdot)$ are uniform in the definitions of Riemannian differentiability and hc-differentiability, $o(\cdot)$ is uniform (both in the case of d_2- and ρ-distortion).

Thus, claim **I** is proved.

Step XI. Using the results of all previous steps, we can consider the mapping η^{-1} (that is, the extension of ξ^{-1}) from $\ker D\psi \cap \operatorname{Box}_2(0, r(1 + o(1)))$ to $\psi^{-1}(t) \cap \operatorname{Box}_2(0,r)$ (see Steps **VII**–**IX**). The established properties of η imply that the measure of $\psi^{-1}(t) \cap \operatorname{Box}_2(0,r)$ equals that of $\ker D\psi \cap \operatorname{Box}_2(0,r)$ up to a factor of $1 + o(1)$, that is, it equals $C(1 + o(1))r^{\nu - \tilde{\nu}}$, where C is obtained above.

Indeed, η is a bi-Lipschitz C^1-mapping since it is a projection of a C^1-surface in a nontangent direction. Consequently, the inverse mapping η^{-1} is also a bi-Lipschitz C^1-mapping. Moreover, since it is differentiable, it is also metrically differentiable, and its metric differential equals $1 + o(1)$ on every direction. The C^1-mapping η^{-1} satisfies $\mathcal{H}^{N-\tilde{N}}(\eta^{-1}(A)) = \mathcal{J}(\eta^{-1}, 0) \cdot \mathcal{H}^{N-\tilde{N}}(A) \cdot (1 + o(1))$, where $A = \eta(\psi^{-1}(t) \cap \operatorname{Box}_2(0,r))$. We can calculate the Jacobian of η^{-1} via its metric differential [Ki, Km1, Km4, Km3, Km5]:

$$\mathcal{J}(\eta^{-1}, 0) = \sigma_{N-\tilde{N}} \Big[\int_{\mathbb{S}^{N-\tilde{N}-1}} (1 + o(1))^{\tilde{N}-N} \, d\mathcal{H}^{N-\tilde{N}}(u) \Big]^{-1} = (1 + o(1)).$$

Thus, $\mathcal{H}^{N-\tilde{N}}(\eta^{-1}(A)) = \mathcal{H}^{N-\tilde{N}}(A)(1 + o(1))$. By taking into account a remark in Step **V** substep **(ii)** of the proof of Theorem 3.7 concerning regular points, we infer $\mathcal{H}^{N-\tilde{N}}(A) = \prod_{k=1}^{M} \omega_{n_k - \tilde{n}_k} \cdot \frac{\mathcal{D}(D\varphi(x))}{\mathcal{D}(\widehat{D}\varphi(x))} r^{\nu - \tilde{\nu}}(1 + o(1))$.

Step XII. Consider the restriction of θ_x to $\psi^{-1}(t) \cap \operatorname{Box}_2(0,r)$. Observe that $\theta_x(\psi^{-1}(t) \cap \operatorname{Box}_2(0,r)) = \varphi^{-1}(t) \cap \operatorname{Box}_2(x,r)$.

The $\mathcal{H}^{N-\tilde{N}}$-measure distortion under this mapping equals $\widetilde{\mathcal{D}}(g|_{\ker D\varphi(x)})$, and $c_{\text{Riem}} = \widetilde{\mathcal{D}}(g|_{\ker D\varphi(x)})$.

Claim **II** is proved.

All $o(\cdot)$ in both claims **I** and **II** are uniform in the radius r and $x = \theta_x(0)$, $x \in \mathcal{U} \Subset \mathbb{M}$.

The proof of the theorem is complete. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Definition 3.12. The (spherical) Hausdorff measure of a set $A \subset \varphi^{-1}(t)$ (constructed with respect to a sub-Riemannian (quasi)metric d and sub-Riemannian balls B_d in d) equals

$$\mathcal{H}^{\nu-\tilde{\nu}}(A) = \omega_{\nu-\tilde{\nu}} \lim_{\delta \to 0} \inf \Big\{ \sum_{i \in \mathbb{N}} r_i^{\nu-\tilde{\nu}} : \bigcup_{i \in \mathbb{N}} B_d(x_i, r_i) \supset A, x_i \in A, r_i \le \delta, i \in \mathbb{N} \Big\}.$$

Property 3.13 ([KmVod])**.** The quasimetric d_2 and the metric d_{cc} (see Definitions 2.9 and 2.11) are locally equivalent.

Lemma 3.14. *Given a set $A \subset \varphi^{-1}(t)$ of $\mathcal{H}^{N-\widetilde{N}}$-measure zero consisting of regular points and $\varepsilon > 0$, there exists a covering of A by the "balls" $\{\mathrm{Box}_2(x_j, r_j) \cap \varphi^{-1}(t)\}_{j \in \mathbb{N}}$, where $x_j \in \varphi^{-1}(t)$, $j \in \mathbb{N}$, the sum of whose $\mathcal{H}^{N-\widetilde{N}}$-measures is less than ε.*

Proof. Fix $\varepsilon > 0$. Represent A as the union of some subsets $A_l \subset A$, $l \in \mathbb{N}$, lying at positive distances from the set $\chi \cap \varphi^{-1}(t)$. Without loss of generality we consider a set A_l instead of A. Suppose that A_l satisfies the stated conditions. Observe that for this set there exists a collection of "balls" $\{B(x_j, r_j) \cap \varphi^{-1}(t)\}_{j \in \mathbb{N}}$, where $x_j \in \varphi^{-1}(t)$ and $B(x_j, r_j)$ are Riemannian balls, $j \in \mathbb{N}$, the sum of whose $\mathcal{H}^{N-\widetilde{N}}$-measures is less than $\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is determined by ε (we specify the exact expression for $\tilde{\varepsilon}$ below). Fix $j \in \mathbb{N}$. For $B(x_j, r_j) \cap \varphi^{-1}(t)$, there exists a collection

$$\{ B_{cc}(x, r) \cap \varphi^{-1}(t) : x \in A_l \cap \varphi^{-1}(t), B_{cc}(x, r) \cap \varphi^{-1}(t) \subset B(x_j, r_j), r > 0 \}.$$

By the $5r$-covering lemma (see [Fe2] for instance) there exists a countable family of disjoint "balls" $\{B_{cc}(x_{j_k}, r_{j_k}) \cap \varphi^{-1}(t)\}$ such that

$$\bigcup_{k \in \mathbb{N}} B_{cc}(x_{j_k}, 5r_{j_k}) \supset A_l \cap B(x_j, r_j).$$

(Here we use the metric d_{cc} since the $5r$-covering lemma is established for metrics, and we cannot say for sure whether it holds for quasimetrics). Since d_2 and d_{cc} are locally equivalent, there exist some constants $0 < C_1, C_2 < \infty$ such that $\mathrm{Box}_2(x, C_1 r) \subset B_{cc}(x, r) \subset \mathrm{Box}_2(x, C_2 r)$ for sufficiently small $r > 0$ and x in some sufficiently small neighborhood. Consequently, there exists a disjoint collection $\{\mathrm{Box}_2(x_{j_k}, C_1 r_{j_k})\}$ such that

$$\bigcup_{k \in \mathbb{N}} \mathrm{Box}_2(x_{j_k}, 5C_2 r_{j_k}) \supset A_l \cap B(x_j, r_j).$$

This implies that

$$\sum_{k \in \mathbb{N}} \mathcal{H}^{N-\widetilde{N}}(\mathrm{Box}_2(x_{j_k}, 5C_2 r_{j_k})) \leq C(5, C_1, C_2) \sum_{k \in \mathbb{N}} \mathcal{H}^{N-\widetilde{N}}(\mathrm{Box}_2(x_{j_k}, C_1 r_{j_k}))$$

$$\leq C(5, C_1, C_2) \mathcal{H}^{N-\widetilde{N}}(B(x_j, r_j) \cap \varphi^{-1}(t)).$$

Then $\{\mathrm{Box}_2(x_{j_k}, 5C_2 r_{j_k})\}_{j,k \in \mathbb{N}}$ is a reqired collection, and the sum of the $\mathcal{H}^{N-\widetilde{N}}$-measures of these "balls" is at most $C(5, C_1, C_2)\tilde{\varepsilon} < \varepsilon$. $\square$

Corollary 3.15. *For each $\mathcal{H}^{N-\widetilde{N}}$-measure zero subset of $\varphi^{-1}(t)$ consisting of regular points we have $\mathcal{H}^{\nu - \tilde{\nu}} = 0$.*

The proof follows directly from the definition of an $\mathcal{H}^{\nu - \tilde{\nu}}$-negligible set.

Lemma 3.16. *Consider a regular point $x \in \mathbb{M}$. Then given a sufficiently small $r > 0$ with $\mathrm{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x)) \cap \chi = \varnothing$ and $\varepsilon > 0$, there exists a covering of $\mathrm{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x))$ by the sets $\mathrm{Box}_2(x_i, r_i) \cap \varphi^{-1}(\varphi(x))$ with $x_i \in \varphi^{-1}(\varphi(x))$ and $\mathrm{Box}_2(x_i, r_i) \subset \mathrm{Box}_2(x, r)$, $i \in \mathbb{N}$, such that*

$$\sum_{i \in \mathbb{N}} \mathcal{H}^{N-\widetilde{N}}(\mathrm{Box}_2(x_i, r_i) \cap \varphi^{-1}(\varphi(x))) < \mathcal{H}^{N-\widetilde{N}}(\mathrm{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x))) + \varepsilon.$$

Proof. Without loss of generality we assume that $r > 0$ satisfies

$$\mathrm{dist}(\mathrm{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x)), \chi) > 0.$$

Since the measure $\mathcal{H}^{N-\widetilde{N}}$ is doubling on $\mathrm{Box}_2(y, s) \cap \varphi^{-1}(\varphi(x))$ for $y \in \varphi^{-1}(\varphi(x))$ and sufficiently small s, in view of the Vitali covering theorem there exists a collection of "balls" $\{\mathrm{Box}_2(x_{i_j}, r_{i_j}) \cap \varphi^{-1}(\varphi(x))\}_{j \in \mathbb{N}}$ with $x_{i_j} \in \varphi^{-1}(\varphi(x))$ and $\mathrm{Box}_2(x_{i_j}, r_{i_j}) \cap \varphi^{-1}(\varphi(x)) \subset \mathrm{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x))$, $j \in \mathbb{N}$, such that

$$\mathcal{H}^{N-\widetilde{N}}(\mathrm{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x))) = \sum_{j \in \mathbb{N}} \mathcal{H}^{N-\widetilde{N}}(\mathrm{Box}_2(x_{i_j}, r_{i_j}) \cap \varphi^{-1}(\varphi(x))).$$

By Lemma 3.14, the $\mathcal{H}^{N-\widetilde{N}}$-negligible set

$$\mathrm{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x)) \setminus \bigcup_{j \in \mathbb{N}} \mathrm{Box}_2(x_{i_j}, r_{i_j}) \cap \varphi^{-1}(\varphi(x)),$$

admits a covering by the collection $\{\mathrm{Box}_2(x_{i_k}, r_{i_k}) \cap \varphi^{-1}(\varphi(x))\}$ of "balls" the sum of whose $\mathcal{H}^{N-\widetilde{N}}$-measures is less than the given $\varepsilon > 0$. The proof is complete. $\square$

Recall that $D_{\mu_1}\mu_2(y)$ stands for the derivative of a measure μ_2 with respect to a measure μ_1 at y:

$$D_{\mu_1}\mu_2(y) = \lim_{r\to 0} \frac{\mu_2(B(y,r))}{\mu_1(B(y,r))}.$$

Theorem 3.17 (The Lebesgue Differentiation of Measures on Level Sets).
The Hausdorff measure $\mathcal{H}^{\nu-\tilde{\nu}}$ of $\mathrm{Box}_2(x,r) \cap \varphi^{-1}(\varphi(x))$, where x is a regular point, and $\mathrm{dist}(\mathrm{Box}_2(x,r) \cap \varphi^{-1}(\varphi(x)), \chi) > 0$, asymptotically equals $\omega_{\nu-\tilde{\nu}}r^{\nu-\tilde{\nu}}$:

$$\mathcal{H}^{\nu-\tilde{\nu}}(\mathrm{Box}_2(x,r) \cap \varphi^{-1}(\varphi(x))) = \omega_{\nu-\tilde{\nu}}r^{\nu-\tilde{\nu}}(1 + o(1)).$$

The derivative $D_{\mathcal{H}^{N-\tilde{N}}}\mathcal{H}^{\nu-\tilde{\nu}}(x)$ equals

$$\frac{1}{\widetilde{\mathcal{D}}(g|_{\ker D\varphi(x)})} \cdot \frac{\omega_{\nu-\tilde{\nu}}}{\prod\limits_{k=1}^{M} \omega_{n_k-\tilde{n}_k}} \cdot \frac{\mathcal{D}(\widehat{D}\varphi(x))}{\mathcal{D}(D\varphi(x))}.$$

Proof. Consider the intersection of the sub-Riemannian ball of a sufficiently small radius $r > 0$ centered at x and the level set $\varphi^{-1}(\varphi(x))$. Fix $\delta > 0$ and some covering $\{\mathrm{Box}_2(y_i, r_i)\}_{i\in\mathbb{N}}$ of this intersection by sub-Riemannian balls as in the definition of the set function $\mathcal{H}^{\nu-\tilde{\nu}}_\delta$. Recall that for a (quasi)metric d we have

$$\mathcal{H}^{\nu-\tilde{\nu}}_\delta(A) = \omega_{\nu-\tilde{\nu}}\inf\left\{\sum_{i\in\mathbb{N}} r_i^{\nu-\tilde{\nu}} : \bigcup_{i\in\mathbb{N}} B_d(x_i, r_i) \supset A, x_i \in A, r_i \leq \delta, i \in \mathbb{N}\right\}$$

Then, setting

$$\alpha(y) = \lim_{r\to 0} \frac{\omega_{\nu-\tilde{\nu}}r^{\nu-\tilde{\nu}}}{\mathcal{H}^{N-\tilde{N}}(\mathrm{Box}_2(y,r) \cap \varphi^{-1}(\varphi(y)))},$$

by Theorem 3.11, we have

$$\omega_{\nu-\tilde{\nu}}\sum_{i\in\mathbb{N}} r_i^{\nu-\tilde{\nu}} = \sum_{i\in\mathbb{N}}[\alpha(y_i) + \delta(y_i, r_i)]\mathcal{H}^{N-\tilde{N}}(\mathrm{Box}_2(y_i, r_i) \cap \varphi^{-1}(\varphi(x)))$$

$$= [\alpha(x) + \Delta(x,r)]\sum_{i\in\mathbb{N}}\mathcal{H}^{N-\tilde{N}}(\mathrm{Box}_2(y_i, r_i) \cap \varphi^{-1}(\varphi(x))),$$

where $\delta(y_i, r_i) \to 0$ as $r_i \to 0$ uniformly in $i \in \mathbb{N}$, and $\Delta(x,r) \to 0$ as $r \to 0$, since g, $D\varphi$ and $\widehat{D}\varphi$ are continuous, and by Theorem 3.11 as well. For the

41

fixed x and $\varepsilon > 0$, we can choose a sufficiently small radius $r > 0$ to satisfy $|\Delta(x, r)| < \varepsilon \cdot \alpha(x)$. Consequently,

$$(1 - \varepsilon)\alpha(x) \sum_{i \in \mathbb{N}} \mathcal{H}^{N-\widetilde{N}}(\text{Box}_2(y_i, r_i) \cap \varphi^{-1}(\varphi(x)))$$

$$\leq \omega_{\nu - \tilde{\nu}} \sum_{i \in \mathbb{N}} r_i^{\nu - \tilde{\nu}} \leq (1 + \varepsilon)\alpha(x) \sum_{i \in \mathbb{N}} \mathcal{H}^{N-\widetilde{N}}(\text{Box}_2(y_i, r_i) \cap \varphi^{-1}(\varphi(x))).$$

Note that for each $\delta > 0$ if the covering in the definition of $\mathcal{H}_\delta^{\nu - \tilde{\nu}}$ is nearly "optimal" (i. e., $\sum_{i \in \mathbb{N}} r_i^{\nu - \tilde{\nu}}$ is nearly minimal), then $\sum_{i \in \mathbb{N}} \mathcal{H}^{N-\widetilde{N}}(\text{Box}_2(y_i, r_i) \cap \varphi^{-1}(\varphi(x)))$ is also nearly minimal. Since the $\mathcal{H}^{N-\widetilde{N}}$-measure is countably additive, by Lemma 3.16 the infimum of the values of the last sum equals $\mathcal{H}^{N-\widetilde{N}}(\text{Box}_2(x, r))$.

Hence,

$$\mathcal{H}^{\nu - \tilde{\nu}}(\text{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x))) = \omega_{\nu - \tilde{\nu}} r^{\nu - \tilde{\nu}}(1 + o(1)),$$

where $o(1) \to 0$ as $r \to 0$, and

$$D_{\mathcal{H}^{N-\widetilde{N}}} \mathcal{H}^{\nu - \tilde{\nu}}(x) = \alpha(x) = \lim_{r \to 0} \frac{\omega_{\nu - \tilde{\nu}} r^{\nu - \tilde{\nu}}}{\mathcal{H}^{N-\widetilde{N}}(\text{Box}_2(x, r) \cap \varphi^{-1}(\varphi(x)))} \qquad (3.10)$$

$$= \frac{1}{\widetilde{\mathcal{D}}(g|_{\ker D\varphi(x)})} \cdot \frac{\omega_{\nu - \tilde{\nu}}}{\prod_{k=1}^{M} \omega_{n_k - \tilde{n}_k}} \cdot \frac{\mathcal{D}(\widehat{D}\varphi(x))}{\mathcal{D}(D\varphi(x))}. \qquad (3.11)$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

The last result motivates the next definition.

Definition 3.18. The *sub-Riemannian coarea factor* equals

$$\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x) = \mathcal{D}(\widehat{D}\varphi(x)) \cdot \frac{\omega_N}{\omega_\nu} \frac{\omega_{\tilde{\nu}}}{\omega_{\widetilde{N}}} \frac{\omega_{\nu - \tilde{\nu}}}{\prod_{k=1}^{M} \omega_{n_k - \tilde{n}_k}}.$$

Remark 3.19. By (3.10), we have

$$\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x) = \mathcal{J}_{\widetilde{N}}(\varphi, x) \cdot D_{\mathcal{H}^{N-\widetilde{N}}} \mathcal{H}^{\nu - \tilde{\nu}}(x) \cdot \frac{\widetilde{\mathcal{D}}(g(x))}{\widetilde{\mathcal{D}}(\tilde{g}(\varphi(x)))} \cdot \frac{\omega_N}{\omega_\nu} \frac{\omega_{\tilde{\nu}}}{\omega_{\widetilde{N}}}$$

since

$$\mathcal{J}_{\widetilde{N}}(\varphi, x) = \mathcal{D}(D\varphi(x)) \cdot \frac{\widetilde{\mathcal{D}}(\tilde{g}(\varphi(x)))\widetilde{\mathcal{D}}(g|_{\ker D\varphi(x)})}{\widetilde{\mathcal{D}}(g(x))}.$$

Corollary 3.20. *Theorem 3.17, Corollary 3.15, and the Lebesgue differentiation theorem (see, for instance, [VodU]) imply that every measurable subset of a regular set $A \subset (\mathbb{D} \cap \varphi^{-1}(t))$, $t \in \widetilde{\mathbb{M}}$, satisfies*

$$\mathcal{H}^{\nu - \tilde{\nu}}(A) = \int\limits_A D_{\mathcal{H}^{N - \tilde{N}}} \mathcal{H}^{\nu - \tilde{\nu}}(x) \, d\mathcal{H}^{N - \widetilde{N}}(x).$$

Theorem 3.21 (Local coarea formula for the set $\mathbb{D}$). *If $\varphi \in C^1(\mathbb{M}, \widetilde{\mathbb{M}})$ then*

$$\int\limits_{\mathbb{D}} \mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x) \, d\mathcal{H}^\nu(x) = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t) \cap \mathbb{D}} d\mathcal{H}^{\nu - \tilde{\nu}}(u).$$

Proof. The Riemannian coarea formula [Fe2] yields

$$\int\limits_{\mathbb{D}} \mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x) \, d\mathcal{H}^\nu(x) = \int\limits_{\mathbb{D}} \frac{\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x)}{\mathcal{D}(g(x))} \, d\mathcal{H}^N(x)$$

$$= \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\widetilde{N}}(t) \int\limits_{\varphi^{-1}(t) \cap \mathbb{D}} \frac{\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, u)}{\mathcal{J}_{\widetilde{N}}(\varphi, u) \widetilde{\mathcal{D}}(g(u))} \, d\mathcal{H}^{N - \widetilde{N}}(u).$$

Calculating the measure derivatives in $\widetilde{\mathbb{M}}$, we obtain

$$\int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\widetilde{N}}(t) \int\limits_{\varphi^{-1}(t) \cap \mathbb{D}} \frac{\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, u)}{\mathcal{J}_{\widetilde{N}}(\varphi, u) \widetilde{\mathcal{D}}(g(u))} \, d\mathcal{H}^{N - \widetilde{N}}(u)$$

$$= \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t) \cap \mathbb{D}} \frac{\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, u) \widetilde{\mathcal{D}}(\tilde{g}(t))}{\mathcal{J}_{\widetilde{N}}(\varphi, u) \widetilde{\mathcal{D}}(g(u))} \, d\mathcal{H}^{N - \widetilde{N}}(u).$$

Finally, taking Remark 3.19 and Corollary 3.20 into account, we infer that

$$\int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t) \cap \mathbb{D}} \frac{\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, u) \widetilde{\mathcal{D}}(\tilde{g}(t))}{\mathcal{J}_{\widetilde{N}}(\varphi, u) \widetilde{\mathcal{D}}(g(u))} \, d\mathcal{H}^{N - \widetilde{N}}(u)$$

$$= \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t) \cap \mathbb{D}} d\mathcal{H}^{\nu - \tilde{\nu}}(u).$$

The proof is complete. $\square$

4   The Characteristic Set

The goal of this section is to prove

Theorem 4.1 (see also [Km7]). *If Assumption 2.26 is fulfilled then*

$$\mathcal{H}^{\nu-\tilde{\nu}}(\chi_t) = 0$$

for almost all $t \in \widetilde{\mathbb{M}}$ (with respect to both $\mathcal{H}^{\tilde{N}}$ and $\mathcal{H}^{\tilde{\nu}}$).

Recall that we denote $\varphi^{-1}(t) \cap \chi$ by χ_t for $t \in \widetilde{\mathbb{M}}$.

We prove the results of this section under the following assumptions.

Assumption 4.2. Assume that the basis vector fields $\{X_i\}_{i=1}^N$ in the preimage are of class C^{M+1}, and $\{\widetilde{X}_j\}_{j=1}^{\widetilde{N}}$ in the image are of class $C^{1,\tilde{\alpha}}$, $\tilde{\alpha} > 0$. Moreover, assume that $\varphi \in C^{M+2}(\mathbb{M}, \widetilde{\mathbb{M}})$, where M is the depth of $\mathbb{M}$ (see Definition 2.1).

Lemma 4.3. *Let $\mathbb{M}$ be a Carnot manifold with $C^{1,\alpha}$-smooth basis vector fields, $\alpha > 0$. From the basis vector fields $\{X_i\}_{i=1}^N$ construct linearly independent vector fields $\{Y_j\}_{j=1}^N$ as $Y_j(x) = \sum_{i=1}^{\dim H_{\deg X_j}} a_{ji}(x) X_i$, where $a_{ji}(x)$ are $C^{1,\omega}$-smooth functions for all i and j, $\omega > 0$. Then the quasimetric of type d_2 constructed with respect to the new basis $\{Y_j\}_{j=1}^N$ is locally equivalent to the initial quasimetric d_2.*

Proof. Denote the quasimetric of type d_2 and the Carnot–Carathéodory metric constructed with respect to the basis $\{Y_j\}_{j=1}^N$ by d_2^Y and d_{cc}^Y respectively. The Ball-Box Theorem of [KmVod, Km9] implies that d_2 and d_{cc} are locally equivalent. Moreover, since the lengths of a curve calculated with respect to different bases are bi-Lipschitz equivalent, the metrics d_2^Y and d_{cc}^Y are bi-Lipschitz equivalent as well. Considering that d_{cc} and d_{cc}^Y are locally equivalent, we infer that so are d_2 and d_2^Y. $\square$

Proof of Theorem 4.1. Observe that the condition $\mathcal{H}^N(\chi) = \mathcal{H}^\nu(\chi) = 0$ and [Fe2, 2.10.25] immediately imply the result.

Assume now that $\mathcal{H}^N(\chi) > 0$.

Step I. Divide χ into finitely many sets, each of which is defined by the structure of $\ker D\varphi$. In other words, represent χ as

$$\chi = \bigcup_{p_1,\dots,p_M} \chi_{p_1\dots p_M},$$

44

where

$$\chi_{p_1\dots p_M} = \Big\{ x \in \chi : \ \dim(H_i(x) \cap \ker D\varphi(x)) = p_i, \ i = 1, \dots, M \Big\}.$$

The assumption $\mathcal{H}^N(\chi) > 0$ implies the existence of a collection $\{p_1, \dots, p_M\}$ with $\mathcal{H}^N(\chi_{p_1\dots p_M}) > 0$ since $\mathcal{H}^N(\chi) \leq \sum\limits_{p_1,\dots,p_M} \mathcal{H}^N(\chi_{p_1\dots p_M})$.

Step II. Fix a collection $\{p_1, \dots, p_M\}$ with this property, the corresponding set $\chi_{p_1\dots p_M}$, and $x \in \chi_{p_1\dots p_M}$ (which is not necessarily a density point). The purpose of Steps **II**–**V** is to prove that

- the set $\chi_{p_1\dots p_M}$ is measurable;

- in the coordinates of the first kind the order of tangency between a level set and its tangent plane is $o(r^M)$ at almost every $x \in \chi_{p_1\dots p_M}$ (with respect to both $\mathcal{H}^N$ and $\mathcal{H}^\nu$),

these facts that are independent of the basis transformations described in Lemma 4.3 on sufficiently small $\mathcal{U} \Subset \mathbb{M}$. Thus, without loss of generality we may assume that at this chosen point x the differential $D\varphi(x)$ is degenerate on basis vector fields $X_{i_1}(x), \dots, X_{i_{N-\widetilde{N}}}(x)$. Consider a compact neighborhood $U \Subset \mathbb{M}$ at whose points y the differential $D\varphi(y)$ does not vanish on the same vector fields as at x.

Step III. On Step **IV**, we are going to prove that $\mathcal{H}^N(\chi_{p_1\dots p_M})$ is measurable. To this end, we now construct a new auxiliary basis $\{Y_j\}_{j=1}^N$ in the same way as in Lemma 4.3. This construction consists of several steps.

(i) The purpose of this substep is to obtain a new horizontal basis the nonzero images of whose vectors under $D\varphi$ at $\chi_{p_1\dots p_M}$ are independent.

The definition of $\chi_{p_1\dots p_M}$ and the structures of $D\varphi$ and $\widehat{D}\varphi$ (see (2.5) and (2.4)) imply that

$$\dim(\ker \widehat{D}\varphi(x) \cap H_1(x)) = \dim(\ker D\varphi(x) \cap H_1(x)) = p_1,$$

where $p_1 > n_1 - \tilde{n}_1$ (see Proposition 2.21, claim **I (a)**). Then, the images of the remaining $n_1 - p_1$ horizontal vectors are linearly independent on $\varphi(U)$.

Next, we consider the matrix of the classical differential $D\varphi$ written with respect to the basis vector fields $X_1, \dots, X_N$ in the preimage and $\widetilde{X}_1, \dots, \widetilde{X}_{\widetilde{N}}$ in the image.

Consider the $\tilde{n}_1 \times n_1$ block of the matrix of $D\varphi$ that corresponds to the horizontal subspaces. Identify its n_1 columns as vectors in $\mathbb{R}^{\tilde{n}_1}$. Then, at x we choose from this collection $n_1 - p_1$ linearly independent vectors $w_1(x), \dots, w_{n_1-p_1}(x)$. At a point $y \in U$ we denote the corresponding column vector as $w_1(y), \dots, w_{n_1-p_1}(y)$. The remaining p_1 vectors $v_1(y), \dots, v_{p_1}(y)$

at $y \in \chi_{p_1...p_{M_1}} \cap U$ are contained in the linear span of the first ones. Consequently, for the set $\chi_{p_1...p_M}$ there exists a transformation $O^{\tilde{n}_1}_{n_1-p_1}(y)$ of $\mathbb{R}^{\tilde{n}_1}$ depending C^{M+1}-smoothly on $y \in U$ such that the images of all $w_1(y),\ldots,w_{n_1-p_1}(y)$ belong to $\mathbb{R}^{n_1-p_1} \times 0^{\tilde{n}_1-(n_1-p_1)}$. To see that this mapping is C^{M+1}-smooth, it suffices to write the coordinates of the linearly independent vectors $\{w_1(y),\ldots,w_{n_1-p_1}(y),e_{i_1},\ldots,e_{i_{\tilde{n}_1-(n_1-p_1)}}\}$ in the standard basis as a matrix and then to calculate the inverse matrix. (Here $\{e_{i_1},\ldots,e_{i_{\tilde{n}_1-(n_1-p_1)}}\}$ are the vectors of the standard basis in $\mathbb{R}^{\tilde{n}_1}$ linearly independent together with $w_1(x),\ldots,w_{n_1-p_1}(x)$; without loss of generality we may assume that the neighborhood U is sufficiently small for the vectors $\{e_{i_1},\ldots,e_{i_{\tilde{n}_1-(n_1-p_1)}}\}$ to be independent together with $w_1(x),\ldots,w_{n_1-p_1}(x)$ for all $y \in U$.) The resulting matrix is the matrix of this transformation. In other words, the matrix

$$(O^{\tilde{n}_1}_{n_1-p_1}(y)[w_1(y)],\ldots,O^{\tilde{n}_1}_{n_1-p_1}(y)[w_{n_1-p_1}(y)])$$

has exactly $\tilde{n}_1 - (n_1 - p_1)$ vanishing columns. Since we can apply to these vectors an orthogonal transformation independent of $y \in U$ without loss of generality we may assume that these vanishing columns have indices from $n_1 - p_1 + 1$ to $\tilde{n}_1$.

By the linear dependence, at the points of $\chi_{p_1...p_M}$ the images of the column vectors $v_1(y),\ldots,v_{p_1}(y)$ also belong to $\mathbb{R}^{n_1-p_1}$. Represent these images (at an arbitrary point $y \in U$) as

$$O^{\tilde{n}_1}_{n_1-p_1}(y)[v_k(y)] = (\omega_k^1(y), \omega_k^2(y))^T,$$

where the dimension of $\omega_k^1(y)$ equals n_1-p_1, and the dimension of $\omega_k^2(y)$ equals $\tilde{n}_1 - (n_1 - p_1)$. Since $O^{\tilde{n}_1}_{n_1-p_1}(y) \in C^M(U)$, where U is a compact neighborhood of the origin, $X_i \in C^{M+1}$, $i = 1,\ldots,N$, it follows that $\omega_k^1(y)$ depends C^M-smoothly on $n_1 - p_1$ columns. Hence, there exist smooth coefficients $b_{k,1}(y),\ldots,b_{k,n_1-p_1}(y)$ with $\omega_k^1(y) = \sum\limits_{l=1}^{n_1-p_1} b_{k,l}(y)w_l(y)$. Then if the columns $w_1(y),\ldots,w_{n_1-p_1}(y)$ correspond to the vectors $X_{i_1}(y),\ldots,X_{i_{n_1-p_1}}(y)$, and a column $v_k(y)$ corresponds to $X_{j_k}(y)$, then upon replacing $X_{j_k}(y)$ by

$$X_{j_k}(y) - \sum_{l=1}^{n_1-p_1} b_{k,l}(y)X_{i_l}(y) \tag{4.1}$$

we find that the part of the new column corresponding to $\omega_k^1(y)$ equals zero, where $k = 1,\ldots,p_1$. Next, since at the points of $\chi_{p_1...p_M}$ the columns $v_1(y),\ldots,v_{p_1}(y)$ belong to $\mathbb{R}^{n_1-p_1}$ we have $\omega_k^2(y) = 0$ on $\chi_{p_1...p_{M_1}}$. Apply the

transformation (4.1) for all $k = 1, \ldots, p_1$. In this case, each new column $v_k(y)$ vanishes at $y \in \chi_{p_1 \ldots p_M}$, $k = 1, \ldots, p_1$.

Thus, we obtain a new horizontal basis the nonzero images of whose vectors under $D\varphi$ at $\chi_{p_1 \ldots p_M}$ are independent.

(ii) Next, we apply similar arguments to the subspace H_2 using the fact that we have already constructed a basis for H_1. Namely, at $y \in \chi_{p_1 \ldots p_M}$, in each of $p_2 - p_1$ vector fields, we delete the parts depending on $n_2 - p_2$ linearly independent vector fields. This yields a new basis for H_2 the nonzero images of whose nonhorizontal vector fields independent and nonhorizontal (belonging to $\widetilde{H}_2$).

Continuing similarly, we obtain new bases for all H_k, $k = 3, \ldots, M$. Denote these vector fields by $\{Y_j\}_{j=1}^N$.

Step IV. By step **III**, the intersection of $\chi_{p_1 \ldots p_M}$ with the closure of every compact neighborhood $W \Subset U$ is closed. Indeed, take $\{y_l\} \to y$ as $l \to \infty$, where $y_l \in \chi_{p_1 \ldots p_M} \cap \overline{W}$. Since $D\varphi(y_l)$ at each point y_l vanishes on a certain collection of basis vector fields by the definition of $\chi_{p_1 \ldots p_M}$, the continuity of the differential implies that $D\varphi(y)$ vanishes on the same vector fields and is non-degenerate on the rest since $W \Subset U$. Thus, the set $\chi_{p_1 \ldots p_M}$ is measurable.

The measurability of $\chi_{p_1 \ldots p_M}$ implies that almost all its points are density points (both with respect to $\mathcal{H}^N$ with balls in the Riemannian metric and $\mathcal{H}^\nu$ with Box$_2$-balls).

Step V. Estimate now the $\mathcal{H}^{N-\widetilde{N}}$-measure of the intersection of the level set passing through a density point $x \in \chi_{p_1 \ldots p_M}$ of $\chi_{p_1 \ldots p_M}$, and the ball Box$_2(x, r)$. To this end, we use in the image and preimage the normal coordinates with respect to the points $\varphi(x)$ and x, and estimate the order of tangency between this level set and the tangent plane. We denote the resulting composition $\theta_{\varphi(x)}^{-1} \circ \varphi \circ \theta_x$ by ψ.

Evaluate ψ at an arbitrary point $y = \sum\limits_{i=1}^{N-\widetilde{N}} t_i Z_{j_i}(0) = \sum\limits_{i=1}^{N-\widetilde{N}} t_i e_{j_i}$, where $Z_{j_i} = D\theta_x^{-1}[Y_{j_i}]$ are the basis vector fields tangent to the level set $\psi^{-1}(\psi(0))$ at zero. We have [NSW]

$$\psi(y) = \sum_{j=1}^M \Big[\frac{1}{j!} \Big(\sum_{i=1}^{N-\widetilde{N}} t_i Z_{j_i} \Big)^j \psi \Big](0) + o(|y|^M)$$

$$= \sum_{j=2}^M \Big[\frac{1}{j!} \Big(\sum_{i=1}^{N-\widetilde{N}} t_i Z_{j_i} \Big)^j \psi \Big](0) + o(|y|^M),$$

where $o(\cdot)$ is uniform on U. Since 0 is a density point of zeroes of $D\psi Z_{j_i}$, the

result of the action of every differential operator on it vanishes at 0 as well:

$$\Big(\sum_{i=1}^{N-\widetilde{N}} t_i Z_{j_i}\Big)^2 \psi(0) = 0.$$

Similar statements hold for almost all (with respect to both $\mathcal{H}^N$ and $\mathcal{H}^\nu$) points of $\chi_{p_1...p_M}$. Consequently, 0 is a density point of the zeroes of the function

$$f(y) = \Big(\sum_{i=1}^{N-\widetilde{N}} t_i Z_{j_i}\Big)^2 \psi(y).$$

Then we obtain

$$\psi(y) = \sum_{j=3}^{M} \Big[\frac{1}{j!}\Big(\sum_{i=1}^{N-\widetilde{N}} t_i Z_{j_i}\Big)^j \psi\Big](0) + o(|y|^M)$$

$$= \sum_{j=4}^{M} \Big[\frac{1}{j!}\Big(\sum_{i=1}^{N-\widetilde{N}} t_i Z_{j_i}\Big)^j \psi\Big](0) + o(|y|^M)$$

since $\Big(\sum\limits_{i=1}^{N-\widetilde{N}} t_i Z_{j_i}\Big)^3 \psi(0) = 0$.

Applying similar arguments, we arrive at $\psi(y) = o(|y|^M)$. This implies that the tangent plane approximates the level set up to $o(|y|^M)$. Indeed, take y in the same level set as 0, and the orthogonal projection y' of y along $(\ker D\psi(0))^\perp$ to the tangent plane. Then

$$\psi(y) = \psi(y') + o(|y|^M) = \psi(y') + D\psi(y')(y - y') + o(|y - y'|) = \psi(y') + C|y - y'|,$$

and since $|C| \ge C_0 > 0$ uniformly on U, we have $|y - y'| = o(|y|^M)$. Observe that here $o(\cdot)$ is uniform on U. Consequently, Theorem 3.7 yields

$$\mathcal{H}^{N-\widetilde{N}}(\psi^{-1}(\psi(0)) \cap \mathrm{Box}_2(0, r)) = Lr^{\nu-\nu_0},$$

where $0 < L_1 \le L \le L_2 < \infty$ uniformly on U, and

$$\mathcal{H}^{N-\widetilde{N}}(\varphi^{-1}(\varphi(x)) \cap \mathrm{Box}_2(x, r)) = C_x r^{\nu-\nu_0}, \qquad (4.2)$$

$\nu_0 > \widetilde{\nu}$. Here the number ν_0 is defined by the set $\chi_{p_1...p_M}$, independently of x. Since the order of tangency is independent of the basis, the box in (4.2) is taken in the initial quasimetric d_2 which is constructed with respect to the initial basis $\{X_i\}_{i=1}^N$. Moreover, the relation

$$\mathcal{H}^{N-\widetilde{N}}(\varphi^{-1}(\varphi(y)) \cap \mathrm{Box}_2(x, r)) = C_y r^{\nu-\nu_0}$$

with $0 < K_1 \le C_y \le K_2 < \infty$ holds for all points $y \in \chi_{p_1 \ldots p_M} \setminus \Sigma_{\chi_{p_1 \ldots p_M}}$ for some set $\Sigma_{\chi_{p_1 \ldots p_M}}$ with $\mathcal{H}^N(\Sigma_{\chi_{p_1 \ldots p_M}}) = \mathcal{H}^\nu(\Sigma_{\chi_{p_1 \ldots p_M}}) = 0$.

Step VI. Suppose that $z \in \widetilde{\mathbb{M}}$. Estimate the measure

$$\mathcal{H}^{\nu - \widetilde{\nu}}(\varphi^{-1}(z) \cap (\chi_{p_1 \ldots p_M} \setminus \Sigma_{\chi_{p_1 \ldots p_M}}) \cap W)$$

for $W \Subset U$. Since d_2 and d_{cc} are locally equivalent, the conditions of $5r$-covering lemma are fulfilled for the balls centered at the points of $\chi_{p_1 \ldots p_M} \setminus \Sigma_{\chi_{p_1 \ldots p_M}}$. Consider a covering $\bigcup\limits_{i \in \mathbb{N}} \mathrm{Box}_2(x_i, r_i)$ of the set

$$\varphi^{-1}(z) \cap (\chi_{p_1 \ldots p_M} \setminus \Sigma_{\chi_{p_1 \ldots p_M}}) \cap W$$

such that the balls $\{\mathrm{Box}_2(x_i, r_i/5l)\}_{i \in \mathbb{N}}$ are disjoint. Here the number l depends on the constants in the equivalence of d_2 and d_{cc}. Then we have

$$\begin{aligned}
\mathcal{H}^{\nu - \widetilde{\nu}}_\delta(\varphi^{-1}&(z) \cap (\chi_{p_1 \ldots p_M} \setminus \Sigma_{\chi_{p_1 \ldots p_M}}) \cap W) \\
&\le C \delta^{\nu_0 - \widetilde{\nu}} \sum_{i \in \mathbb{N}} r_i^{\nu - \nu_0} \le C_1 \delta^{\nu_0 - \widetilde{\nu}} \sum_{i \in \mathbb{N}} \mathcal{H}^{N - \widetilde{N}}(\varphi^{-1}(\varphi(x)) \cap \mathrm{Box}_2(x_i, r_i)) \\
&\le C_2 \delta^{\nu_0 - \widetilde{\nu}} \sum_{i \in \mathbb{N}} \mathcal{H}^{N - \widetilde{N}}(\varphi^{-1}(\varphi(x)) \cap \mathrm{Box}_2(x_i, r_i/5l)) \le \delta^{\nu_0 - \widetilde{\nu}} L \to 0
\end{aligned}$$

as $\delta \to 0$ since $L \le \mathcal{H}^{N - \widetilde{N}}(\varphi^{-1}(z) \cap (\chi_{p_1 \ldots p_M} \setminus \Sigma_{\chi_{p_1 \ldots p_M}}) \cap W)$ is independent of $\delta > 0$. Thus, the intersection of every level set with $\chi_{p_1 \ldots p_M} \setminus \Sigma_{\chi_{p_1 \ldots p_M}}$ has $\mathcal{H}^{\nu - \widetilde{\nu}}$-measure zero. Consequently,

$$\mathcal{H}^{\nu - \widetilde{\nu}}(\varphi^{-1}(z) \cap (\chi \setminus \Sigma)) = 0$$

where $\Sigma = \bigcup\limits_{p_1 \ldots p_M} \Sigma_{\chi_{p_1 \ldots p_M}}$ and $\mathcal{H}^N(\Sigma) = \mathcal{H}^\nu(\Sigma) = 0$. Now the theorem follows from [Fe2, 2.10.25]. □

5   The Degeneration Set

Theorem 5.1. *For $\mathcal{H}^{\widetilde{\nu}}$-almost all $t \in \widetilde{\mathbb{M}}$ we have*

$$\mathcal{H}^{\nu - \widetilde{\nu}}(\varphi^{-1}(t) \cap Z) = 0.$$

Proof. **Step I.** We have $Z = Z_1 \cup Z_2$, where $\mathcal{H}^{\widetilde{N}}(\varphi(Z_1)) = \mathcal{H}^{\widetilde{\nu}}(\varphi(Z_1)) = 0$, and $\mathcal{H}^{N - \widetilde{N}}(\varphi^{-1}(t) \cap Z_2) = 0$ for all $t \in \widetilde{\mathbb{M}}$ (see, for instance, [Km6]).

Step II. Note that $\mathcal{H}^{\widetilde{N}}(\varphi(Z_2)) < \infty$ since otherwise we can represent it as a countable union of sets of finite measure. Next, decompose Z_2 as $\bigcup\limits_{i=0}^{\widetilde{N}-1} C_i$, where $\operatorname{rank} D\varphi(x) = i$ for $x \in C_i$.

Fix $0 \le i \le \widetilde{N} - 1$. Without loss of generality we may assume that C_i is a compact set. Consider a compact set $\widetilde{C}_i \subset C_i$ such that $\varphi(\widetilde{C}_i)$ is a compact subset of the set of density points of $\varphi(C_i)$ and $\mathcal{H}^{\widetilde{N}}(\varphi(C_i) \setminus \varphi(\widetilde{C}_i)) < \varepsilon$ for a fixed arbitrary $\varepsilon > 0$. Consider the mapping

$$\xi_i(x) = (\varphi(x), x_{N-i+1}, \dots, x_N) : \mathbb{R}^N \to \mathbb{R}^{\widetilde{N}} \times \mathbb{R}^{N-i}.$$

Observe that $\xi_i(C_i)$ is an N-rectifiable set. Our assumption implies that ξ_i is locally bi-Lipschitz on C_i. Consequently, $\mathcal{H}^{N-\widetilde{N}}(\xi_i(\varphi^{-1}(t) \cap \widetilde{C}_i)) = 0$ for all $t \in \widetilde{\mathbb{M}}$. Consider the mapping $\pi : \xi(C_i) \to \mathbb{R}^{\widetilde{N}}$ defined as

$$\pi(x_1, \dots, x_{\widetilde{N}+N-i}) = (x_1, \dots, x_{\widetilde{N}}).$$

Apply the classical coarea formula to π, C_i and $\widetilde{C}_i$:

$$\int\limits_{\xi(C_i)} \mathcal{J}(\pi, x) \, d\mathcal{H}^N(x) \ge \int\limits_{\xi(C_i)} \chi_{\xi(\widetilde{C}_i)}(x) \mathcal{J}(\pi, x) \, d\mathcal{H}^N(x)$$

$$= \int\limits_{\varphi(C_i)} d\mathcal{H}^{\widetilde{N}}(t) \int\limits_{\pi^{-1}(t) \cap \xi(C_i)} \chi_{\xi(\widetilde{C}_i)}(u) \, d\mathcal{H}^{N-\widetilde{N}}(u). \quad (5.1)$$

By the definition of $\widetilde{C}_i$, we find that $\mathcal{J}(\pi, x) = 1$ for $x \in \xi(\widetilde{C}_i)$. Furthermore,

$$\mathcal{H}^{N-\widetilde{N}}(\pi^{-1}(t) \cap \xi(C_i)) = \mathcal{H}^{N-\widetilde{N}}(\xi_i(\varphi^{-1}(t) \cap \widetilde{C}_i)) = 0.$$

It follows from (5.1) that $\mathcal{H}^N(\xi(\widetilde{C}_i)) = 0$. Finally, have

$$\mathcal{H}^N(\widetilde{C}_i) = \mathcal{H}^\nu(\widetilde{C}_i) = 0$$

because ξ_i is bi-Lipschitz. Since $\varepsilon > 0$ is arbitrary, there exists a set Σ_i of measure zero in $\varphi(C_i)$ such that

$$\mathcal{H}^N(\varphi^{-1}(\varphi(C_i) \setminus \Sigma_i) \cap C_i) = \mathcal{H}^\nu(\varphi^{-1}(\varphi(C_i) \setminus \Sigma_i) \cap C_i) = 0.$$

Therefore, $\mathcal{H}^\nu\Big(\bigcup\limits_{i=0}^{\widetilde{N}-1} \varphi^{-1}(\varphi(C_i) \setminus \Sigma_i) \cap C_i\Big) = 0.$

Step III. Since φ is contact, it is locally Lipschitz with respect to every sub-Riemannian metric. Reasoning as in [Fe2, Theorem 2.10.25], we obtain

$$0 \le \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t) \cap \bigcup\limits_{i=0}^{\widetilde{N}-1} \varphi^{-1}(\varphi(C_i) \backslash \Sigma_i) \cap C_i} d\mathcal{H}^{\nu - \tilde{\nu}}(u)$$

$$\le C(\mathrm{Lip}(\varphi), \nu, \tilde{\nu}) \mathcal{H}^{\nu} \Big(\bigcup_{i=0}^{\widetilde{N}-1} \varphi^{-1}(\varphi(C_i) \backslash \Sigma_i) \cap C_i \Big) = 0.$$

Furthermore, the $\mathcal{H}^{\widetilde{N}}$-measure zero set $\bigcup\limits_{i=0}^{\widetilde{N}-1} \Sigma_i$ is included into an $\mathcal{H}^{\widetilde{N}}$-measure zero Borel set Σ. Then, its preimage $\varphi^{-1}(\Sigma)$ is also a Borel set, and the set $\varphi^{-1}(\Sigma) \cap Z_2$ is measurable. By the definition,

$$\mathcal{H}^{\widetilde{N}}(\varphi(\varphi^{-1}(\Sigma) \cap Z_2)) = \mathcal{H}^{\tilde{\nu}}(\varphi(\varphi^{-1}(\Sigma) \cap Z_2)) = 0.$$

combining the results, we deduce that

$$\int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t) \cap Z} d\mathcal{H}^{\nu - \tilde{\nu}}(u) = 0,$$

and thus we complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

6   The Coarea Formula

Theorem 6.1 (Coarea Formula)**.** *If a contact mapping $\varphi : \mathbb{M} \to \widetilde{\mathbb{M}}$ satisfies Assumption 2.26 then the coarea formula holds:*

$$\int\limits_{\mathbb{M}} \mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) \, d\mathcal{H}^{\nu}(x) = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t)} d\mathcal{H}^{\nu - \tilde{\nu}}(u). \qquad (6.1)$$

Proof. Obvserve that $\mathbb{M} = \mathbb{D} \sqcup \chi \sqcup Z$. Consequently,

$$\int\limits_{\mathbb{M}} \mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) \, d\mathcal{H}^{\nu}(x) = \int\limits_{\mathbb{D}} \mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) \, d\mathcal{H}^{\nu}(x)$$

$$+ \int\limits_{\chi} \mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) \, d\mathcal{H}^{\nu}(x) + \int\limits_{Z} \mathcal{J}^{SR}_{\widetilde{N}}(\varphi, x) \, d\mathcal{H}^{\nu}(x).$$

Theorem 3.21 yields

$$\int\limits_{\mathbb{D}} \mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x)\, d\mathcal{H}^{\nu}(x) = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t)\cap\mathbb{D}} d\mathcal{H}^{\nu-\tilde{\nu}}(u).$$

Theorem 4.1, implies

$$\int\limits_{\chi} \mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x)\, d\mathcal{H}^{\nu}(x) = 0 = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t)\cap\chi} d\mathcal{H}^{\nu-\tilde{\nu}}(u).$$

Finally, since $\mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x) = 0$ on Z, Theorem 5.1 yields

$$\int\limits_{Z} \mathcal{J}_{\widetilde{N}}^{SR}(\varphi, x)\, d\mathcal{H}^{\nu}(x) = 0 = \int\limits_{\widetilde{\mathbb{M}}} d\mathcal{H}^{\tilde{\nu}}(t) \int\limits_{\varphi^{-1}(t)\cap Z} d\mathcal{H}^{\nu-\tilde{\nu}}(u).$$

Combinig these relations, we obtain (6.1). The theorem follows. $\qquad\square$

References

[AgMa] Agrachev A., Marigo A. *Nonholonomic tangent spaces: intrinsic construction and rigid dimensions*, Electron. Res. Announc. Amer. Math. Soc. **9** (2003), 111–120.

[AmKi] Ambrosio L., Kirchheim B. *Rectifiable sets in metric and Banach spaces*, Math. Ann. **318** (2000), 527–555.

[Be] Bellaïche A. *Tangent Space in Sub-Riemannian Geometry*, In: Sub-Riemannian geometry. Basel, Birkhauser Verlag, 1996, 1–78.

[BLU] Bonfiglioli A., Lanconelli E., Uguzzoni F. *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer, 2007.

[1] Chow W. L., *Über Systeme von linearen partiellen Differentialgleichungen erster Ordung*, Math. Ann. **117** (1939), 98-105.

[CS] Citti G., Sarti A. *A cortical based model of perceptual completion in the roto-translation space*, Lecture Notes of Seminario Interdisciplinare di Matematica **3** (2004), 145–161.

[EG] Evans L. C., Gariepy R. F. *Measure theory and fine properties of functions*, CRC Press, Boca Raton, 1992.

[Fe1]   Federer H. *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[Fe2]   Federer H. *Geometric Measure Theory*, NY: Springer, 1969.

[FeFl]   Federer H., Fleming W. H. *Normal and Integral Currents*, Ann. Math. **72** (2) (1960), 458-520.

[FoSt]   *Folland G. B., Stein E. M.* Hardy spaces on homogeneous groups, Princeton Univ. Press, 1982.

[GMS]   Giaquinta M., Modica G., Souček J. *Cartesian currents in the calculus of variations*, V. I, II, Springer-Verlag, Berlin, 1998.

[Gr]   Gromov M. *Carnot–Carathéodory Spaces Seen From Within*, In: Sub-Riemannian geometry. Basel, Birkhauser Verlag, 1996, 79–318.

[He]   Heinonen J. *Calculus on Carnot groups*, In: Fall school in analysis (Jyväskylä, 1994). Jyväskylä, University of Jyväskylä, 1994, 1-32.

[HlPa]   Hladky R. K., Pauls S. D. *Minimal surfaces in the roto-translation group with applications to a neuro-biological image completion model*, arXiv:math.DG/0509636, 27 Sep. 2005.

[Je]   Jean F. *Uniform estimation of sub-riemannian balls*, Journal on Dynamical and Controle Systems **7** (4) (2001), 473–500.

[Km1]   Karmanova M. B. *Metric Differentiability of Mappings and Geometric Measure Theory*, Doklady Mathematics **71** (2) (2005), 224–227.

[Km2]   Karmanova M. B. *Rectifiable Sets and Coarea Formula for Metric-Valued Mappings*, Doklady Mathematics, **73** (3) (2006), 323–327.

[Km3]   Karmanova M. B. *Metric Rademacher Theorem and Area Formula for Metric-Valued Lipschitz Mappings*, Vestnik NSU **6** (4) (2006), 50–69.

[Km4]   Karmanova M. *Formulas of Geometric Measure Theory on Rectifiable Metric Spaces*, Contemporary Mathematics **424** (2007), 103–136.

[Km5]   Karmanova M. B. *Area and Coarea Formulas for Sobolev Mappings with Values in a Metric Space*, Siberian Mathematical Journal **48** (4) (2007), 778–788.

[Km6]   Karmanova M. *Rectifiable Sets and Coarea Formula for Metric-Valued Mappings*, Journal of Functional Analysis **254** (5) (2008), 1410–1447.

[Km7]   Karmanova M. B. *Characteristic Set of Smooth Mappings of Carnot Manifolds*, Dokl. AN **425** (3) (2009), 314–319.

[Km8]   Karmanova M. B. *A New Approach to Investigation of Carmot–Carathéodory Geometry*, Doklady Mathematics, **82** (2) (2010), 746–750.

[Km9]   Karmanova M. *The New Approach to Investigation of Carnot–Carathéodory Geometry*, GAFA (2011), accepted.

[Km10]   Karmanova M. B. *Convergence of Scaled Vector Fields and Local Approximation Theorem on Carnot–Carathéodory Spaces and Applications*, Dokl. AN **440** (6) (2011), 736–742.

[KmVod] Karmanova M., Vodopyanov S. *Geometry of Carnot–Carathéodory Spaces, Differentiability, Coarea and Area Formulas*, In: Analysis and Mathematical Physics. Birkhäuser, 2009. P. 233–335.

[Ki]   Kirchheim B. *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure*, Proc. AMS. **121** (1994), 113–123.

[Kr]   Kronrod A. S. *On functions of two variables* (in Russian), Uspehi Matem. Nauk (N. S.) **5** (1950), 24–134.

[LiYa]   F. Lin and X. Yang, *Geometric measure theory*, Science Press, Beijing a. o., 2002.

[Mg1]   Magnani V. *Elements of Geometric Measure Theory on sub-Riemannian groups*, Tesi di Perfezionamento. Pisa: Scuola Normale Superiore (Thesis), 2002.

[Mg2]   Magnani V. *Blow-up of regular submanifolds in Heisenberg groups and applications*, Cent. Eur. J. Math. **4** (1) (2006), 82–109.

[MrMo1] Margulis G. A., Mostow G. D. *The differential of quasi-conformal mapping of a Carnot–Carathéodory space*, Geometric and Functional Analysis **5** (2) (1995), 402–433.

[MrMo2] Margulis G. A., Mostow G. D. *Some remarks on the definition of tangent cones in a Carnot-Carathéodory space*, Journal D′Analyse Math. **80** (2000), 299–317.

[Mn] Montgomery R. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, Providence, AMS, 2002.

[NSW] Nagel A., Stein E. M., Wainger S. *Balls and metrics defined by vector fields I: Basic properties*, Acta Math. **155** (1985), 103–147.

[Ot] Ohtsuka M. *Area Formula*, Bull. Inst. Math. Acad. Sinica **6** (2) (2) (1978), 599–636.

[Pa] Pansu P. *Geometrie du group d'Heisenberg*, Univ. Paris VII, 1982.

[Po] Postnikov M. M. *Lectures in Geometry. Semester V: Lie Groups and Lie Algebras*, Moscow, "Nauka", 1982.

[2] Rashevsky P. K., *Any two points of a totally nonholonomic space may be connected by an admissible line*, Uch. Zap. Ped. Inst. im. Liebknechta. Ser. Phys. Math. **2** (1938), 83-94.

[RoSt] Rothschild L. P., Stein E.M. *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

[Vod1] Vodopyanov S. K. *The Theory of Lebesgue Integral: Lecture Notes in Calculus*, Novosibirsk State University Press, Novosibirsk, 2006.

[Vod2] Vodop'yanov S. K. *Differentiability of Curves in Carnot Manifold Category*, Dokl. AN **410** (4) (2006), 1–6.

[Vod3] Vodopyanov S. K. *Differentiability of mappings of Carnot Manifolds and Isomorphism of Tangent Cones*, Doklady Mathematics **74** (3) (2006), 844–848.

[Vod4] Vodopyanov S. *Geometry of Carnot–Carathéodory Spaces and Differentiability of Mappings*, Contemporary Mathematics **424** (2007), 247–301.

[Vod5] Vodopyanov S. K. *Differentiability of Mappings in the Geometry of Carnot Manifolds*, Sib. Math. Zh., **48** (2) (2007), 251–271.

[VodG] Vodop'yanov S. K., Greshnov A. V. *On the Differentiability of Mappings of Carnot–Carathéodory Spaces*, Doklady Mathematics **67** (2) (2003), 246–250.

[VodKm1] Vodopyanov S. K., Karmanova M. B. *Local Geometry of Carnot Manifolds Under Minimal Smoothness*, Doklady Mathematics, **75** (2) (2007), 240–246.

[VodKm2] Vodopyanov S. K., Karmanova M. B. *Local Approximation theorem in Carnot Manifolds Under Minimal Assumptions on smoothness*, Doklady Mathematics, **80** (1) (2009), 585–589.

[VodU] Vodopyanov S. K., Ukhlov A. D. *Set functions and their applications in the theory of Lebesgue and Sobolev spaces*, I: Siberian Adv. Math. **14** (4) (2004), 78–125; II: Siberian Adv. Math. **15** (1) (2005), 91–125.